### 1.2 Mathematical language

The mathematical language is much simpler than our natural languages, but in many respects it is similar. One handles symbols, which one might call nouns or names, but which are technically called expressions, and these symbols represent objects. Expressions can be combined, according to specific syntactic rules, to form other expressions. Moreover, expressions can be combined with special symbols to form statements. And statements can be combined to form other statements. To be precise, two different strings of symbols may represent the same statement. Therefore one calls the strings of signs that state something with a specific name; they are called formulas. So a formula makes a statement, but two different formulas may correspond to the same statement, very much the same way as two words may correspond to the same notion.

Usually one uses letters from some alphabet as names of the represented objects. As the number of letters available is small, usually one letter is used as a name of some object only temporarily. Later, on another occasion, the same letter may be used to represent a different object. For instance, in the previous section we used the symbol " $P_{0}$ " to denote a certain value of heating power. Of course, we shall not restrict the use of that symbol to represent that heating power for the rest of our lives. This economy of symbols has the disadvantage that the scope of a symbol has to be declared or made clear somehow. So the author of any mathematical text has to pay attention to that question and make the scope of a symbol clear to the reader. Generally the use of a letter as a name of an object is announced by some introducing sentence. Very often one finds sentences like the following one: " Let $S$ be a ......". Such symbol introducing sentence automatically terminates any previous use of that symbol.

Sometimes a letter may represent an object that is still unspecified. It may be that the type of object is specified but the actual object itself is unknown or may vary depending on other circumstances. In this case the letter is called a variable. If the symbol represents a definite object it is called a constant name ${ }^{I}$.

Frequently two expressions represent the same object. If this is the case one writes the symbol " $=$ " between the two expressions ${ }^{2}$. So the sequence of symbols $a=b$ tells us that the symbols " $a$ " and " $b$ " are two different names of the same object. It does not say that the symbols " $a$ " and " $b$ " are the same! It may seem superfluous to explain the use of the symbol " $=$ ". But a careful analysis of mathematical mistakes committed by university students reveals that many of these mistakes are due to misunderstanding the meaning of the symbol " $=$ ". It is unfortunate that many computer languages (C, FORTRAN and others) use the symbol " $=$ " with a different meaning.
$a=b$ is a statement, or more precisely, a formula. It can be either true or false depending on the objects represented by the symbols " $a$ " and " $b$ ". In some cases one might not be able to know whether it is true or false.

So far we have not introduced numbers and algebraic operations. Nevertheless, we shall use an example of elementary algebra to explain more details of equality. The symbols " 2 " and " 5 " are constant names. Let $x$ be a numeric variable. Then the string of symbols $2+x=5$ is a syntactically correct formula. With the rules of algebra one

[^0]concludes that $x=3$. But this latter sequence of symbols does not mean that now $x$ is a constant name! $x$ is still a variable. Both equalities $2+x=5$ and $x=3$ can be true or false depending on what number $x$ represents. On the other hand one could, for some reason, introduce $x$ as an abbreviation of the number 3. In this case one writes $x \underset{\text { def. }}{=} \quad 3$. Then $x$ is a constant name. The subscript "def." is an explicit declaration that the formula $x=3$ is true. Sometimes one wants to use such abbreviations only temporarily and sometimes they shall be valid for ever. We shall distinguish these cases and write "Def." under the $=$ sign if the abbreviation is valid for ever and "def." if it is a temporary abbreviation.
\[

$$
\begin{equation*}
\text { *** } \underset{\text { Def. }}{=} \quad \cdots \quad \text { means } * * * \text { is a permanent abbreviation of } . . . . \tag{1.2.1}
\end{equation*}
$$

\]

and

$$
\begin{equation*}
*^{* * *} \underset{\text { def. }}{=} \quad \ldots . \quad \text { means } * * * \text { is a temporary abbreviation of } \ldots . . \tag{1.2.2}
\end{equation*}
$$

Accordingly the symbols on the left hand side of the $=$ sign become temporarily or permanently constant names if the right hand side is a constant name. If the right hand side is a variable, the left hand side will continue a variable in either case.
The logical contrary of $a=b$ is written as $a \neq b$. So " $a$ " $\neq " b$ " is always true (unless you need new spectacles), because the symbols are different and the strings with quotation marcs are names of the symbols. But $a=b$ may be true or false depending on the objects represented by the symbols " $a$ " and " $b$ ". The statement $a \neq a$ is always false.

The construction of $a \neq b$ starting from $a=b$ is an example of formation of formulas from other formulas. $a \neq b$ is the negation of the original formula. In general the negation of a formula $\Phi$ will be written as $\neg \Phi$. So $\neg a=b$ and $a \neq b$ make the same statement. The negation of a formula is defined by its truth values: $\neg \Phi$ is true if and only if $\Phi$ is false and $\neg \Phi$ is false if and only if $\Phi$ is true. One may show these cases in a table of truth values:

| $\Phi$ | $\neg \Phi$ |
| :--- | :--- |
| true | false |
| false | true |

Table 1.2.1 Truth values of negation
Other constructions of new formulae from old ones combine two formulae to form a third one: If $\Phi$ and $\Psi$ are formulae then $\Phi \vee \Psi, \Phi \wedge \psi, \Phi \Rightarrow \Psi$ and $\Phi \Leftrightarrow \Psi$ are also formulae. Table 1.2.2 gives the spoken English names of these formulae and the truth values that define them.

Table 2.2.2 English names and truth values of combined formulae.

| $\Phi$ | $\Psi$ | $\Phi \vee \Psi$ <br> $\Phi$ or $\Psi$ | $\Phi \wedge \psi$ <br> $\Phi$ and $\Psi$ | $\Phi \Rightarrow \Psi$ <br> $\Phi$ implies $\Psi$ | $\Phi$ and $\Psi$ are <br> equivalent |
| :---: | :---: | :---: | :---: | :---: | :---: |
| false | false | false | false | true | true |
| false | true | true | false | true | false |
| true | false | true | false | false | false |
| true | true | true | true | true | true |

It would have been enough to introduce only $\Phi \vee \Psi$ and the negation. The statement $\Phi \wedge \psi$ is in fact the same as $\neg(\neg \Phi \vee \neg \Psi)$ and $\Phi \Rightarrow \Psi$ is the same as $\Psi \vee \neg \Phi$ (show this!). Finally $\Phi \Leftrightarrow \Psi$ is the same as $(\Phi \Rightarrow \Psi) \wedge(\Psi \Rightarrow \Phi)$. Note that here parentheses were used as additional symbols of mathematical language. In the case of $\neg(\neg \Phi \vee \neg \Psi)$ they tell the reader that the leftmost negation acts on the formula $\neg \Phi \vee \neg \Psi$ and not only on the formula $\neg \Phi$. The polish mathematician Jan Łukasiewicz ${ }^{3}$ invented a notation that does not need parenthesis. In this notation, which is known as polish notation, the symbols " $\vee$ ", " $\wedge$ ", " $\Rightarrow$ ", " $\Leftrightarrow$ " are written not between the two formulae that they combine but in front of them. So, for instance, $\neg(\neg \Phi \vee \neg \Psi)$ would have the form $\neg \vee \neg \Phi \neg \Psi$, whereas $(\neg \neg \Phi) \vee \neg \Psi$ would be written as $\vee \neg \neg \Phi \neg \Psi$. This is an extremely useful notation in formal studies of mathematical language, but for most people it is a bit hard to read and to understand. Therefore we shall use the conventional notation, which makes the introduction of parenthesis necessary.

There exist combinations of formulae that result in a formula that is always true. For instance the formula $(\Phi \wedge(\Phi \Rightarrow \Psi)) \Rightarrow \Psi$ is always true independent of what the formulae $\Phi$ and $\Psi$ might be. Such combined formulae that are always true independent of the nature of the formulae that appear in the combination are called tautologies. There also exist formulae that are always false, for instance $(\neg \Psi) \wedge(\Phi \wedge(\Phi \Rightarrow \Psi))$ or $\Phi \wedge \neg \Phi$. These are called contradictions ${ }^{4}$.

One says that the symbols " $a$ " and " $b$ " appear in the formula $a=b$. The symbol " $a$ " appears in the formula $a=a$. Whether $a=b$ is true or false depends on which objects are represented by " $a$ " and " $b$ ". Despite the fact that " $a$ " appears in $a=a$, this formula is always true. Then we say that $a=a$ is true for all $a$. This will be written as a new formula, which is derived from $a=a$ using a special symbol; we shall write it as $\forall a: a=a$.

Let $\Phi$ be a formula. If the symbol " $a$ " designates a variable then one can form a new formula

[^1]\[

$$
\begin{equation*}
\forall a: \Phi \tag{1.2.3}
\end{equation*}
$$

\]

The meaning of this new formula is: "The formula $\Phi$ is true for all $a$ ". This new formula may be true or false. For instance the formula $\forall a: a=a$ is true, but the formula $\forall a: a=b$ is false because, whatever the symbol " $b$ " may stand for, there is always some $a$ different from the object $b$.
The symbol " $\forall$ " is a logical quantifier. Apart from this all-quantifier it is convenient to introduce an existence-quantifier " $\exists$ ". The meaning of this can be defined in terms of the all-quantifier and negation by saying that

$$
\begin{equation*}
\exists a: \Phi \quad \text { is an abbreviation of } \quad \neg \forall a: \neg \Phi \tag{1.2.4}
\end{equation*}
$$

In the same spirit of $\underset{\text { Def. }}{=}$ and $\underset{\text { def. }}{=} \quad$ we shall write the statement (1.2.4) formally as

$$
\begin{equation*}
\exists a: \Phi \quad \underset{\text { Def. }}{\Leftrightarrow} \quad \neg \forall a: \neg \Phi \tag{1.2.5}
\end{equation*}
$$

In English one pronounces " $\exists a: \Phi$ " as: "there exists an $a$ such that $\Phi$ is true". Such formula can be true or false. For instance, $\exists a: a=b$ is true and $\exists a: a \neq a$ is false. As one advances in pure mathematics one recognizes that mathematical existence may be a very formal matter. Sometimes the existence of an object is affirmed that one will never be able to get hold of and Martin Heidegger would have good reasons to criticize such existences.

The symbol " $a$ " appears in the formulas $a=b$ and $\forall a: a=b$. But the appearance of " $a$ " is totally different in these two cases. In both formulas $a$ is a variable. In $a=b$ we may substitute this variable by a constant name and this substitution results in a new and different formula. For instance, we may substitute " $a$ " by "Albert Einstein". This latter sequence of symbols is a name of a definite object, which is also known as the inventor of the theory of relativity. So it is a constant name. Depending on what " $b$ " stands for, the resulting statement Albert Einstein $=b$ may be true or false, but it is a meaningful statement. On the other hand if we substitute $a$ by a constant name in $\forall a: a=b$ we get simply a sequence of symbols that does not pertain to our mathematical language. For instance, $\forall 5: 5=b$ is not a syntactically correct formula. The same situation prevails with formulae containing existence-quantifiers.

An appearance of a string of symbols that represents a constant name or can be substituted by a constant name is called a free appearance. An appearance of a string of symbols that cannot be substituted by a constant name is called a bound appearance. Formulas without freely appearing variables are called closed formulas.

Here we discussed only appearances in formulas, but later we shall see that free and bound appearances can also refer to expressions. For example " $x$ " appears in the expression $\int_{1}^{2} \ln (x) d x$ in bound form and in $\ln (x)$ it appears freely. Variables that appear in bound form are often called dummy variables.

A letter that appears in bound form can be substituted by any other letter that does not appear otherwise, and this substitution does not change the statement. So, for example, $\forall a: a=b$ and $\forall x: x=b$ are exactly the same statement. But we may not replace " $a$ " by " $b$ ", this would change the statement. The scope of a bound variable is automatically
limited to the formula that follows immediately after its introduction through the quantifier. So in a long sequence of symbols ........... $\exists x:(x=b \vee x=c)_{\uparrow} \ldots \ldots .$. . the scope of " $x$ " starts at the first arrow and ends at the second one. It would be perfectly legal to use " $x$ " again in the rest, which is denoted with dots, and there " $x$ " would have a different meaning. However, it is wise to avoid this in order to facilitate legibility. It is easy to avoid such double occurrence of letters because the bound letter can be changed. So if one wants to use an " $x$ " in the symbols, which we abbreviated with the dots, one had better write the part between the arrows in the form $\exists y:(y=b \vee y=c)$.

The scope of bound variables is especially important when a statement contains several logical quantifiers. The scope of " $y$ " in the formula

$$
\begin{equation*}
\exists x_{\uparrow} \forall y: \Phi_{\uparrow} \tag{1.2.6}
\end{equation*}
$$

is the range indicated by the two arrows. " $x$ " is being introduced by means of the existence quantifier outside this range, where " $y$ " has no meaning. Therefore the existing object $x$ cannot depend on $y$. It serves for all $y$. On the other hand in the statement

$$
\begin{equation*}
\forall y \exists x: \Phi \tag{1.2.7}
\end{equation*}
$$

the symbol " $x$ " is introduced within the scope of " $y$ " and therefore it may be that every object $y$ has got a different object $x$ such that $\Phi$ holds. So $x$ may depend on $y$. To make this clear one sometimes writes formulae such as (1.2.7) also in the form

$$
\begin{equation*}
\forall y \exists x_{y}: \Phi \tag{1.2.8}
\end{equation*}
$$

The statements (1.2.7) and (1.2.8) are the same, the index " $y$ ", is not necessary, but it may help the reader.

So far we have explained very basic notions. Even the simple sign " $=$ " has been found worthy to be explained. If you try you will discover that the more basic a notion is, more difficult it becomes to explain it. Now one may ask: what do "true" and "false" mean? As long as one stays in pure manipulation of strings of symbols one may answer: "They do not mean anything. They are just two more symbols that are associated to formulas." The situation will however change once we use the mathematical language to describe something. Then "true" and "false" will gain meanings. At least these signs are related to the meaning of " $=$ ". Attributing "false" to the statement $a=a$ is quite different from attributing "true" to that statement. If $\Phi[A]$ is a formula in which an expression $A$ appears freely and $\Phi[A \nearrow B]$ a formula that one obtains from $\Phi[A]$ by substituting $B$ in some of the places where $A$ appears one has the right to draw the following conclusions:

$$
\begin{equation*}
\text { If } \Phi[A] \text { is true and } \Phi[A \nearrow B] \text { is false then conclude that } A \neq B \text { is true. } \tag{1.2.9}
\end{equation*}
$$

and

$$
\text { If } A=B \text { is true and } \Phi[A] \text { is true, then conclude that } \Phi[A \nearrow B] \text { is true. (1.2.10) }
$$

Further, when one constructs a mathematical theory, one postulates that certain basic formulas are true. The truth of these basic statements has a defining meaning. These statements are part of the definition of the theory. They are called the axioms of the theory. If one then applies such a mathematical theory to the real world the "true" and
"false" get further concrete meaning. If, for instance, the theory predicts that a building will withstand an earthquake of a given magnitude a "false" of such statement has a very definite and dramatic meaning.

From the axioms of a mathematical theory one can deduce the truth values of other formulas according to certain rules. If the formulae $\Phi_{1}, \Phi_{2}, \ldots, \Phi_{k}$ are known to be true one has the right to conclude that $\Psi$ is true if $\left(\Phi_{1} \wedge \Phi_{2} \wedge \ldots . \wedge \Phi_{k}\right) \Rightarrow \Psi$ is a tautology. So for instance, if we know that $\Phi$ is true and that $\Phi \Rightarrow \Psi$ is true, we have the right to conclude that $\Psi$ is true. Such deductions of truth, which ultimately start from the axioms of a theory, are called a proof. By "ultimately" we mean that one could start from the axioms, but in practice one will very often start from formulas that have previously been proven starting from the axioms. The proofs will inform us about the truth of formulas. If the system of axioms of a theory is consistent it should never occur that a formula $\Phi$ as well is its negation $\neg \Phi$ follow from the axioms.

If one wants to prove a formula that has the form $\Phi \Rightarrow \Psi$ one may start with the assumption that $\Phi$ is true because if $\Phi$ happened to be false the formula $\Phi \Rightarrow \Psi$ would be true anyway. But one must not start with the assumption that $\Psi$ is true! This is a mistake very often committed by beginners. $\Psi$ might be false and from a false statement one can correctly deduce anything even true statements. Such deduction will not show that $\Phi \Rightarrow \Psi$ is true. What can be assumed is that $\Psi$ is false. If one is able to deduce from this assumption and from the axioms that $\Phi$ is false too one has actually shown that $\Phi \Rightarrow \Psi$ is true. This is so because $(\neg \Psi \Rightarrow \neg \Phi) \Rightarrow(\Phi \Rightarrow \Psi)$ is a tautology.
In order to prove formulas that have the form $\forall a: \Phi$ one starts "let $a$ be given arbitrarily" and in the remainder one must not assume any thing special about $a$. In order to prove formulas like $\exists a: \Phi$ one may try to find or construct an $a$ that satisfies $\Phi$. This constructive sort of prove of existence is the most valuable one. But sometimes it turns out to be impossible. Then one might show that the assumption that $\forall a: \neg \Phi$ is true leads to a contradiction.

Naively one might expect that the provable formulas are exactly the true ones. But this is not the case. Kurt Friedrich Gödel ${ }^{5}$ showed in 1931 with his famous Incompleteness Theorem I that a language sufficiently rich to formulate arithmetic cannot have any consistent rules of deduction that permit to prove all closed true formulas of arithmetic. ${ }^{6}$ That means there are always closed formulas $\Phi$ such that there exists no proof that shows that $\Phi$ is true neither is there a proof that $\neg \Phi$ is true. One of the two statements, $\Phi$ or $\neg \Phi$, is supposedly true and the other one is false, but we shall never know which one. In his second Incompleteness Theorem Gödel further showed that within a given theory with the language of that theory it cannot be shown that the system of axioms is consistent. These theorems tell us that human cognition has very profound limitations.

[^2]
[^0]:    ${ }^{1}$ In text books of mathematical logic such symbols are simply called "constants". But in physics and other quantitative sciences the word "constant" is also used with a different meaning. Therefore we shall use the term "constant name".
    ${ }^{2}$ The physician and mathematician Robert Recorde introduced the symbol " $=$ " in 1557.

[^1]:    ${ }^{3}$ * 21 December 1878, $\dagger 13$ February 1956
    4 The word "contradiction" is frequently used in the so called "dialectic logic". The reader had better steer well clear of this mental quagmire! As an example we cite Hegel: "Identity is the identity of identity and non-identity." An early precursor of that sort of logic is due to Feliciano Silva. Miguel de Cervantes cites him to explain the sources of madness of Don Quijote: "La razón de la sinrazón que a mi razón se hace, de tal maneira mi razón enflaquece, que com razón me quejo de la vuestra fermosura".

[^2]:    * April 28, 1906, † January 14, 1978
    ${ }^{6}$ For more details see Joseph R. Shoenfield: Mathematical Logic Addison-Wesley Publishing Company (1967) ISBN 0-201-07028-6.

