### 1.4 Relations and Mappings

There are two more axioms to be introduced into set theory. But the axioms stated so far permit us to discuss a considerable number of important facts and we postpone the remaining axioms to a point where they can be duly appreciated.
Let us introduce a new kind of mental object. For any given objects $a$ and $b$ (they may also be equal) we can again form a totality of these objects, but this time the order of objects shall be important. This sort of totality is called an ordered pair and it is written as $\langle a, b\rangle$. As usual, when one invents a new type of object a comment concerning the meaning of " $=$ " is necessary:

$$
\begin{equation*}
\forall a \forall b \forall x \forall y:(\langle a, b\rangle=\langle x, y\rangle \underset{\text { Def. }}{\Leftrightarrow} \quad(a=x \wedge b=y)) \tag{1.4.1}
\end{equation*}
$$

If $A$ and $B$ are sets, intuitively one would expect that the totality of all ordered pairs $\langle a, b\rangle$ with $a \in A$ and $b \in B$ is a set. But this fact should be deduced from the axioms. We will only give the main ideas and leave the details as an exercise: First one uses the pairing axiom, the union axiom, the powerset axiom and the subset axiom to form a set of all sets of the form $\{\{a\},\{a, b\}\}$ with $a \in A$ and $b \in B$ :

$$
\begin{equation*}
C \underset{\text { def. }}{=}\{x \in \mathrm{P}(\mathrm{P}(A \cup B)) \mid \exists(a \in A) \exists(b \in B): x=\{\{a\},\{a, b\}\}\} \tag{1.4.2}
\end{equation*}
$$

Next one defines for any ordered pair the left and right component:

$$
\begin{equation*}
\forall a \forall b:(\mathrm{L}[\langle a, b\rangle] \underset{\text { def. }}{=} \quad a) \text { and } \forall a \forall b:(\mathrm{R}[\langle a, b\rangle] \underset{\text { def. }}{=} \quad b) \tag{1.4.3}
\end{equation*}
$$

and for the variable $c \in C$, with $C$ defined by (1.4.2), and variable $p$ that represents ordered pairs one writes the formula

$$
\begin{equation*}
c=\{\{\mathrm{L}[p]\},\{\mathrm{L}[p], \mathrm{R}[p]\}\} \tag{1.4.4}
\end{equation*}
$$

Next one shows that for any $c \in C$ there exists a unique ordered pair $p$ such that the formula (1.4.4) is true. The existence is simple. In order to show the uniqueness one will have to investigate several cases and use the theorems 1.3 .2 and 1.3.3. Finally one applies the replacement axiom and completes the proof.
The set of ordered pairs $\langle a, b\rangle$ with $a \in A$ and $b \in B$ is called the Cartesian product of $A$ and $B$ and it is written as $A \times B$;

$$
\begin{equation*}
A \times B=\{\langle a, b\rangle \mid a \in A \wedge b \in B\} \tag{1.4.5}
\end{equation*}
$$

The Cartesian product of sets is the basis of several very important notions in mathematics and its applications in quantitative sciences. Fist of all, the notion of relation can be defined with Cartesian products. A relation between objects from a set $A$ and objects from a set $B$ can be viewed as a subset of the Cartesian product $A \times B$. Let $R$ be a subset of $A \times B$. An object $a$ from $A$ is said to fulfill the relation $R$ with an object $b$ from $B$ if and only if the ordered pair $\langle a, b\rangle$ is an element of $R$. And this is usually written putting the symbol " $R$ " between the symbols " $a$ " and " $b$ "; ${ }^{1}$

$$
\begin{equation*}
\forall(a \in A) \forall(b \in B):(a R b \underset{\text { Def. }}{\Leftrightarrow}\langle a, b\rangle \in R) \tag{1.4.6}
\end{equation*}
$$

[^0]Let us see how this definition relates to intuitive ideas about relation. Example1: Let $A$ be the set of all women that live in a certain city and $B$ the set of all men living in that city. The relation "is married to" determines a subset of the set $A \times B$ of ordered pairs that contains all the married couples in that city (always ladies coming first in the couple). This subset is the extensional presentation of the relation "is married to" in the imagined city. Example 2: We have not yet introduced numbers. But of course, all readers believe to know what a number is and what is meant by the relation $a<b$ ( $a$ is smaller than $b$ ). Let $A=B$ and $B=\{1,2,3,4,5\}$. The Cartesian product set $A \times B$ can be symbolically represented by a rectangular arrangement of points in a plane. Figure 1.4.1. shows this arrangement and it shows the subset $<$.


Fig. 1.4.1
Representation of the Cartesian product $A \times B$. The set $<$ is marked.

There are many types of relations frequently used in sciences. Certain relations defined in a single set are especially important. With defined in a single set we mean that the two sets of the Cartesian product are the same (as in our second example). A partial order is a type of relation frequently uncounted: Partial orders are characterized by the following properties:
Definition: A relation $\prec \subset M \times M$ is called a partial order on $M$ if and only if it has the following properties:
Reflexivity:

$$
\begin{equation*}
\forall(a \in M): \quad a \prec a \tag{1.4.7}
\end{equation*}
$$

Antisymmetry

$$
\begin{equation*}
\forall(a \in M) \forall(b \in M):((a \prec b \wedge b \prec a) \Rightarrow a=b) \tag{1.4.8}
\end{equation*}
$$

Transitivity: $\quad \forall(a \in M) \forall(b \in M) \forall(c \in M):((a \prec b \wedge b \prec c) \Rightarrow a \prec c)$
If a partial order relation fulfills also the following condition

$$
\begin{equation*}
\forall(a \in M) \forall(b \in M):(a \prec b \quad \vee \quad b \prec a \vee a=b) \tag{1.4.10}
\end{equation*}
$$

it is called a total order.

Definition: A relation $\sim \subset M \times M$ is called an equivalence relation on $\boldsymbol{M}$ if and only if it has the following properties:
Reflexivity:

$$
\begin{equation*}
\forall(a \in M): \quad a \sim a \tag{1.4.11}
\end{equation*}
$$

Symmetry:

$$
\begin{equation*}
\forall(a \in M) \forall(b \in M):(a \sim b \Rightarrow b \sim a) \tag{1.4.12}
\end{equation*}
$$

Transitivity: $\quad \forall(a \in M) \forall(b \in M) \forall(c \in M):((a \sim b \wedge b \sim c) \Rightarrow a \sim c)$
An equivalence relation on a set $M$ naturally decomposes this set into disjoint subsets of equivalent elements. Let $M$ be a set with an equivalence relation $\sim \subset M \times M$. For any $a \in M$ one can define the subset

$$
\begin{equation*}
c_{a} \underset{\text { def. }}{=} \quad\{x \in M \mid x \sim a\} \tag{1.4.14}
\end{equation*}
$$

These sets are called equivalence classes.
Theorem 1.4.1: The collection of classes

$$
\begin{equation*}
C_{\sim} \underset{\text { def. }}{=}\{D \in \mathrm{P}(M) \mid \exists d \in M: D=\{x \in M \mid x \sim d\}\} \tag{1.4.15}
\end{equation*}
$$

has the following properties:

$$
\begin{gather*}
\forall\left(D \in C_{\sim}\right) \forall(a \in D) \forall(b \in D): a \sim b  \tag{1.4.16}\\
\forall\left(D \in C_{\sim}\right) \forall\left(E \in C_{\sim}\right):(D \neq E \Rightarrow D \cap E=\varnothing)  \tag{1.4.17}\\
\cup C_{\sim}=M \tag{1.4.18}
\end{gather*}
$$

The proof of the statements (1.4.16), (1.4.17), and (1.4.18) is left as an exercise.

Now we come back to the general case and we permit that the two factors of the Cartesian product may be different.
Definition: A relation $F \subset A \times B$ is called mapping or function if it satisfied the following condition:

$$
\begin{equation*}
\forall(a \in A) \exists!(b \in B): a F b \tag{1.4.19}
\end{equation*}
$$

So any $a \in A$ determines a unique element in $B$, which is also written as $F(a)$ and is called "the value of $F$ at the point $a$ " or the "image of $a$ ". Generally when dealing with a function $F$, one does not use the notation $a F b$ to express that $\langle a, b\rangle \in F$. Instead one writes an equality $b=F(a)$.

The word mapping is quite adequate for that sort of relation. A geographic map establishes a relation that associates to any point of a geographic region a point on a piece of paper. If $F \subset A \times B$ is a mapping one writes $F: A \rightarrow B$ and this symbolic statement is spoken as " $F$ maps $A$ into $B$ ". $A$ is called the domain of $\boldsymbol{F}$ and $B$ the co-domain.of $\boldsymbol{F}$. It is not necessary that all elements of $B$ appear as values of $F$. The set of values that do appear is called the range or image of $\boldsymbol{F}$ and this set is written as $\operatorname{ran}(F)$ :

$$
\begin{equation*}
\text { for } F: A \rightarrow B \quad \text { define } \quad \operatorname{ran}(F) \underset{\text { Def. }}{=} \quad\{b \in B \mid \exists(a \in A): b=F(a)\} \tag{1.4.20}
\end{equation*}
$$

If $\operatorname{ran}(F)=B$ the function $F$ is said to be surjective ${ }^{2}$ and one says " $F$ maps $A$ onto B".

$$
\begin{equation*}
\text { for } F: A \rightarrow B \quad, \quad F \text { is surjective } \underset{\text { Def. }}{\Leftrightarrow} \operatorname{ran}(F)=B \tag{1.4.21}
\end{equation*}
$$

In general several different elements $a, b, \ldots$ in the domain may have the same image $F(a)=F(b)$. If the function is such that this never occurs the function is said to be injective.

$$
\begin{align*}
& \text { for } F: A \rightarrow B \quad, \\
& F \text { is injective } \quad \underset{\text { Def. }}{\Leftrightarrow} \quad \forall\left(a_{1} \in A\right) \forall\left(a_{2} \in A\right):\left(F\left(a_{1}\right)=F\left(a_{2}\right) \Rightarrow a_{1}=a_{2}\right) \tag{1.4.22}
\end{align*}
$$

A function that is surjective and injective is said to be bijective or it is called abijection.

$$
\begin{align*}
& \text { for } F: A \rightarrow B \quad, \\
& F \text { is bijective } \quad \underset{\text { Def. }}{\Leftrightarrow} \quad(F \text { is injective } \wedge F \text { is surjective }) \tag{1.4.23}
\end{align*}
$$

The bijective mappings are so important that we shall introduce a special symbol that represents the statement that $F$ maps $A$ bijectively onto $B$.

$$
\begin{equation*}
F: A \leftrightarrow B \underset{\text { Def. }}{\Leftrightarrow} F \text { maps } A \text { bijectively onto } B \tag{1.4.24}
\end{equation*}
$$

Definition: For any relation $R \subset A \times B$ the inverse of $R$ is defined as the set

$$
\begin{equation*}
\text { For } R \subset A \times B \text { define : } \quad R^{-1} \underset{\text { Def. }}{=} \quad\{\langle b, a\rangle \in B \times A \mid \quad\langle a, b\rangle \in R\} \tag{1.4.25}
\end{equation*}
$$

Exercise: Determine the inverse of the relation " $<$ " of example 2 and show the corresponding set in figure 1.4.1. Use the graphical representation of Cartesian products (the one of example 2) and work out examples of $\operatorname{ran}(F)$, surjective mappings, injective mappings and bijective mappings.

Now the inverse of a relation becomes especially important in the case of functions. The following theorem makes this connection:

Theorem 1.4.2: Let $F: A \rightarrow B$ be a function. The inverse relation $F^{-1}$ is also a function if and only if $F$ is bijective.

The proof is left as an exercise.
The following definitions are frequently used:
Let $F: A \rightarrow B$ be a function and $\tilde{A} \subset A$ and $\tilde{B} \subset B$ subsets. Then one defines the following sets:

[^1]\[

$$
\begin{gather*}
F(\tilde{A}) \underset{\text { Def. }}{=}\{b \in B \mid \exists(a \in \tilde{A}): b=F(a)\}  \tag{1.4.26}\\
F^{-1}(\tilde{B}) \underset{\text { Def. }}{=}\{a \in A \mid \exists(b \in \tilde{B}): b=F(a)\} \tag{1.4.27}
\end{gather*}
$$
\]

$F(\tilde{A})$ is called the image of $\tilde{A}$ and $F^{-1}(\tilde{B})$ is called the inverse image of $\tilde{B}$. The function $F$ determines a function $\quad F_{I \tilde{A}}: \tilde{A} \rightarrow B$ such that $\forall(a \in \tilde{A}): F_{\mid \tilde{A}}(a)=F(a)$. This function is called the restriction of $F$ to the set $\tilde{A}$. Further let $G: B \rightarrow C$ be another function. Then one defines the concatenated function $G \circ F: A \rightarrow C$ such that

$$
\begin{equation*}
\forall(a \in A): G \circ F(a)=G(F(a)) \tag{1.4.28}
\end{equation*}
$$

For any set $M$ we define the identity function $\mathbf{i}_{M}: M \rightarrow M$ such that $\forall(a \in M): \mathbf{i}_{M}(a)=a$. If $F: A \rightarrow B$ is a bijection then one has

$$
\begin{equation*}
F^{-1} \circ F=\mathbf{i}_{A} \tag{1.4.29}
\end{equation*}
$$

and

$$
\begin{equation*}
F \circ F^{-1}=\mathbf{i}_{B} \tag{1.4.30}
\end{equation*}
$$

Further we urge the beginner to prove the following theorems:
Theorem 1.4.3: Let $A, B$ and $C$ be sets. If $F: A \leftrightarrow B$ and $G: B \leftrightarrow C$ then $G \circ F: A \leftrightarrow C$.

Theorem 1.4.4: Let $M$ be a set and $\Xi \subset P(M) \times P(M)$ the following relation:

$$
\begin{equation*}
\forall(A \in \mathrm{P}(M)) \forall(B \in \mathrm{P}(M)):(A \Xi B \underset{\text { def. }}{\Leftrightarrow} \exists F:(F: A \leftrightarrow B)) \tag{1.4.31}
\end{equation*}
$$

Then $\Xi$ is an equivalence relation.
Exercise: Determine the equivalence classes of $\Xi$ for the case $M=\{1,2,3,4\}$.
Finally we mention a generalization of relation which has some relevance in natural languages. The notion of ordered pair can obviously be generalized permitting more than two elements in the collection. So for instance, one may form triples $\langle a, b, c\rangle$, quadruples $\langle a, b, c, d\rangle$ or generally $n$-tuples $\left\langle a_{1}, a_{2}, \ldots \ldots . . a_{n}\right\rangle$. An elegant way of defining these objects makes use of the function concept. Let $I$ be a set of indices, for instance, $I=\{1,2, \ldots, n\}$ and suppose we have a set $A_{i}$ for any $i \in I$. Then we may define an $n$ tuple to be a function $F: I \rightarrow \bigcup_{i \in I} A_{i}$ such that for all $i \in I$ one has $F(i) \in A_{i}$. The Cartesian product of the family of sets is the set of all $n$-tuples:

$$
\begin{equation*}
\underset{i \in I}{\times} A_{i} \underset{\text { def. }}{=}\left\{F: I \rightarrow \bigcup_{i \in I} A_{i} \mid \forall(i \in I): F(i) \in A_{i}\right\} \tag{1.4.32}
\end{equation*}
$$

An $n$-ary relation is a subset of $\underset{i \in I}{ } A_{i}$. Further one defines a 1-nary relation on a set $A$ to be a subset of $A$.
The use of $n$-ary relations in mathematics with $n>2$ is seldom. But in natural languages these relations are very common. Think of the following sentence; "Yesterday George transferred 100 Dollars to my account." One may look at this sentence in a somewhat different way: "Yesterday ... transferred" may be considered a subset of triples of objects. The triple $\langle$ George, 100 Dollars, my account〉 is an element of this set if the statement is true. "Yesterday ... transferred" is a 3-ary relation. Natural languages are complicated because they are very flexible. "Yesterday ... transferred" could also be a binary (2-ary) relation. We may also pronounce the correct sentence: "Yesterday George transferred 100 Dollars." There the same "Yesterday ... transferred" works as a binary relation. In mathematics the role of every element in an $n$-tuple is determined by its place or by the index $i$. In natural languages the role of the objects are expressed by word order and/or cases (in Latin: nominative, dative, accusative and ablative and in Greek: ONOMA $\Sigma$ TIKH, $\Delta \Omega$ TIKH and AITIATIKH) and/or prepositions. Hungarian has got 18 cases. So Hungarians can form $n$-ary relations with pretty high $n$.


[^0]:    ${ }^{1}$ In polish notation one would write $R$ in front of $a$ and $b$.

[^1]:    ${ }^{2}$ This term is of French origin.

