1.5 Natural numbers and set theory II

Many scientists and engineers think that theory is not worth anything as long as it does not come up with concrete numbers. There might be a lot of truth in such judgment; nevertheless it is intimately related to two wrong ideas. First the idea that values of physical quantities are numbers, and secondly, that numbers are concrete objects. The notion of number is the paragon of abstraction.

What do we mean by saying "there are three glasses on the table"? Three is some<u>thing</u>[1] that the set $\{\text{Pierre Curie, Max Planck, Michael Faraday}\}$ has in common with the set $\{\text{Danube, Tiber, Thames}\}$. Well, but these sets have in common the fact that they consist of European elements and this fact is not the number three. To exclude such unwanted properties one might take more and more sets with three elements so that the only remaining common property is the number three. But there are two points to be taken care of: 1) No matter how many different sets with three elements we take they all have in common the fact that they are sets and this is not the number three. So the number three cannot solely be characterized as being a common property of certain sets. One has to add that this property also distinguishes these sets from other sets. 2) Of course it is not acceptable to define the number three by a procedure that takes sets with <u>three</u> elements, because this is a circular argument. The selection of sets has to be defined without reference to the number that one wants to define. This can be done with the help of bijections. Obviously all sets with the same number of elements can be mapped bijectively onto each other. So one may define the number three to be the property that all sets that can be mapped bijectively onto the set $\{\text{Danube, Tiber, Thames}\}$ have in common and that distinguishes these sets from any other set.

This is abstraction. The word abstraction comes from the Latin verb *abstrahere*, which means to take away, to withdraw, or to remove. We remove the fact that Danube, Tiber and Thames are rivers, that they are located in Europe and whatever fact we may imagine that is not the wanted evaluation of the size of this set. The size of a set A that is defined this way is called its ***cardinal number*** and it is written as $|A|$. Two sets have the same cardinal number if and only if they can be mapped bijectively onto each other.

$$\forall(\text{set } M)\forall(\text{set } N):\Big(|M|=|N| \underset{Def.}{\Leftrightarrow} \exists F:(F:M\leftrightarrow N)\Big) \tag{1.5.1}$$

Cantor used this idea for judging the size of certain infinite sets. For the time being we shall be concerned only with finite sets.

One may object to this construction of a notion by means of abstraction. Two points remain dubious: Will there be anything left taking unwanted things away? And if so, is what is left unique? For instance two sets M and N that can be mapped bijectively onto each other also have the property that their powersets $\mathsf{P}(M)$ and $\mathsf{P}(N)$ can be mapped bijectively onto each other. One may say that this is not really a different property. At least from an extensional point of view the existence of a bijection $F:M\leftrightarrow N$ is equivalent to the existence of a bijection $G:\mathsf{P}(M)\leftrightarrow\mathsf{P}(N)$. The condition $\exists G:(G:\mathsf{P}(M)\leftrightarrow\mathsf{P}(N))$ will judge the same sets as equally sized as the

[1] Is three a thing? The German word "etwas" would be much better than the word "something".

condition $\exists F:(F:M \leftrightarrow N)$. But it is preferable to appoint a definite mental object as a number. That can be done:

Definition: The cardinal number $|M|$ of a set M is any of the sets N that satisfy $\exists F:(F:M \leftrightarrow N)$ together with a mark that tells us that this new object is no longer a set and the equality of theses objects is no longer defined by the extensionality axiom but by the condition (1.5.1).

Similar procedures can be used to define many other abstract notions.

The usual constant names

$$\begin{aligned} 0 &\underset{Def.}{=} |\varnothing| \\ 1 &\underset{Def.}{=} |\{\text{Johann Sebastian Bach}\}| \\ 2 &\underset{Def.}{=} |\{\text{Albert Einstein, Max Planck}\}| \\ 3 &\underset{Def.}{=} |\{\text{Danube, Tiber, Thames}\}| \\ &\;\;. \\ &\;\;. \end{aligned} \tag{1.5.2}$$

can be defined. The choice of the sets with one, two, three etc. elements is arbitrary. But this is inconvenient because it turns the task to talk about all possible numbers a difficult one. It is better to select a *standard representative* among the sets N that satisfy $\exists F:(F:M \leftrightarrow N)$. Let us write the standard representative of a number n with the symbol $\frown{n}$. One has $|\frown{n}| = n$. The numbers 0, 1, 2, … are not sets, but their standard representatives are sets.

There is only one set with cardinality zero, which is $\varnothing$. This has to be the standard representative of that number. So $\frown{0} = \varnothing$. The other standard representatives should be defined in such a way that they can be constructed systematically. This can be done with the help of the following "elevator operation": For any set A we define a set by adding the set itself as a new element:

$$\text{for set } A \text{ define:} \qquad A^{\uparrow} \underset{Def.}{=} A \cup \{A\} \tag{1.5.3}$$

We shall call $A^{\uparrow}$ the elevation of A. Because of theorem 1.3.1 this operation really adds a new element to the original set.

Exercise: Prove the following theorem:

Theorem 1.5.1: $\forall(\text{set } A)\forall(\text{set } B):\left(A^{\uparrow} = B^{\uparrow} \quad \Rightarrow \quad A = B\right)$

We may generate all standard representatives by applying the elevator operation successively beginning with the representative $\frown{0}$. This way we get

$$\begin{array}{lll} \frown{0} & = & \varnothing \\ \frown{1} & \underset{Def.}{=} & \{\varnothing\} \\ \frown{2} & \underset{Def.}{=} & \{\varnothing,\{\varnothing\}\} \\ \frown{3} & \underset{Def.}{=} & \{\varnothing,\{\varnothing\},\{\varnothing,\{\varnothing\}\}\} \\ & . & \\ & . & \end{array} \tag{1.5.4}$$

So $\overset{\frown}{1} = \overset{\frown}{0}^{\uparrow}$, $\overset{\frown}{2} = \overset{\frown}{1}^{\uparrow}$, $\overset{\frown}{3} = \overset{\frown}{2}^{\uparrow}$ and so on. This "and so on" has to be formulated in a precise way.

The elevator operation can be applied endlessly generating more and more standard representatives whose cardinal numbers form more and more numbers. Endlessly means without end or "finis" (Latin end). This means we have here an ***infinity***. The opponents of set theory accepted this sort of infinity defined by a not ending generating process. But they did not accept to consider the totality of all numbers that can be generated an existing entity. In fact the expression "that can be generated" is quite a dubious one: obviously a human being can apply the elevator operation only a finite number of times. What is really meant by "that can be generated" is something else, something inherent in the generating process itself without reference to an executing being or machine. Here we shall be willing to accept the totality of standard representatives that can be generated from $\varnothing$ by endless application of the elevator operation as an existing set. This is quite a decision to accept talking about infinite objects! So if we are willing to accept that a set of all standard representatives that can be generated from $\varnothing$ by means of the elevator operation exists then that set of sets should have the property that it contains $\varnothing$ as an element and that the elevation $a^{\uparrow}$ of any element a of that set is again an element of that set. Sets with this property receive a special name. They are called ***inductive sets.***

Definiton: A set A is called ***inductive*** if and only if 1) all its elements are sets, 2) $\varnothing \in A$ and 3) $\forall a : \left(a \in A \Rightarrow a^{\uparrow} \in A \right)$.

The decision to accept infinite sets can now be formulated with the ***infinity axiom*****: There exist inductive sets:**

$$\exists (\text{set } A) : A \text{ is inductive} \tag{1.5.5}$$

Our standard representatives are obviously contained in any inductive set. This allows us to formulate the above "and so on" in a precise way: The infinity axiom allows us to write the sentence: Let M be an inductive set. Next the subset axiom allows us to form the set of all standard representatives of natural numbers:

$$\overset{\frown}{\mathbb{N}} \underset{Def.}{=} \left\{ x \in M \mid \forall (\text{set } N) : (N \text{ is inductive} \Rightarrow x \in N) \right\} \tag{1.5.6}$$

For any x in $\overset{\frown}{\mathbb{N}}$ the cardinal number $|x|$ exists and is uniquely determined by x. Then the replacement axiom permits us to form the set of all cardinal numbers of elements of $\overset{\frown}{\mathbb{N}}$:

$$\mathbb{N} \underset{Def.}{=} \left\{ |x| \;\middle|\; x \in \overset{\frown}{\mathbb{N}} \right\} \tag{1.5.7}$$

This set is called the ***set of natural numbers*** and its elements are called ***natural numbers.***

From the definition (1.5.6) it follows that $\frown{\mathbb{N}}$ is a subset of any inductive set. Then the following important theorem is another immediate consequence:

Theorem 1.5.2: For any subset $A \subset \frown{\mathbb{N}}$ one has: A is inductive $\Rightarrow$ $A = \frown{\mathbb{N}}$

Let us define the mapping $C : \frown{\mathbb{N}} \to \mathbb{N}$ that associates to any element of $\frown{\mathbb{N}}$ its cardinal number:

$$\forall\left(a \in \frown{\mathbb{N}}\right): \quad C(a) \underset{def.}{=} |a| \tag{1.5.8}$$

A first consequence of that theorem is:

Theorem 1.5.3: The mapping $C : \frown{\mathbb{N}} \to \mathbb{N}$ that associates to any element of $\frown{\mathbb{N}}$ its cardinal number is bijective.

Proof: The very definition of $\mathbb{N}$, (1.5.7), means that $C : \frown{\mathbb{N}} \to \mathbb{N}$ is surjective. It remains to show that this mapping is also injective. Let

$$U \underset{def.}{=} \left\{x \in \frown{\mathbb{N}} \,\middle|\, \forall\left(y \in \frown{\mathbb{N}}\right): \left(|y| = |x| \Rightarrow y = x\right)\right\} \tag{1.5.9}$$

We have to show that $U = \frown{\mathbb{N}}$. To that end we use the theorem 1.5.2 and try to show that U is inductive. The uniqueness of $\varnothing$ implies that $\varnothing \in U$. Next we have to show that $\forall(a \in U):\left(a \in U \Rightarrow a^{\uparrow} \in U\right)$ is true. So let $a \in U$ be given arbitrarily. If $a^{\uparrow} \in U$ was false there would be b in $\frown{\mathbb{N}}$ such that $|b| = \left|a^{\uparrow}\right|$ and $b \neq a^{\uparrow}$. If this b were the elevation of some c in $\frown{\mathbb{N}}$ this c would be different from a and it would have the same cardinality as a (exercise: show this!). But as $a \in U$ there is no other set in $\frown{\mathbb{N}}$ with the cardinality $|a|$. Therefore the element b cannot be the elevation of an element of $\frown{\mathbb{N}}$. Then one could take b away from $\frown{\mathbb{N}}$ without destroying the inductivity. So $\frown{\mathbb{N}} \setminus \{b\}$ would still be an inductive set. But that contradicts the fact that $\frown{\mathbb{N}}$ is a subset of any inductive set. So $a^{\uparrow} \in U$ must be true. So we have shown that U is an inductive set. Then the theorem 1.5.2 implies that $U = \frown{\mathbb{N}}$ and this means that $C : \frown{\mathbb{N}} \to \mathbb{N}$ is in fact injective. This completes the proof and at this point we also terminate the scope of the variables a and b and the validity of the definition (1.5.9) so that the symbols a, b, and U may be used for other purposes.

The fact that $C : \frown{\mathbb{N}} \to \mathbb{N}$ is a bijection permits transferring the structure that exists in $\frown{\mathbb{N}}$ to the set of numbers. This goes as follows: For every number n there is a unique standard representative $\frown{n}$, which is given by $\frown{n} = C^{-1}(n)$. On this standard representative we can apply the known set theoretic operations and then we can go back to the set of numbers applying C. First we may define a function $S : \mathbb{N} \to \mathbb{N}$ called the *successor function.*

$$\forall(n \in \mathbb{N}):\left(S(n) \underset{def.}{=} \left|\left(C^{-1}(n)\right)^{\uparrow}\right|\right) \tag{1.5.10}$$

Despite the fact that this successor function is very important, we only used the temporary " $\underset{def.}{=}$ ". There are two reasons for this limitation: later, once we have defined the sum of numbers, we shall write the successor operation in the simple form $n+1$ and further, the letter S is far to often used in physics to occupy its use with the successor operation for ever.

The first application of this definition is a transcription of the theorem 1.5.2 to numbers. The set A that appears in the theorem can be mapped onto a set $C(A)$. This set may be determined by some formula. So it may be the set of numbers n for which a certain formula $\Phi(n)$ is true. Then the theorem 1.5.2 appears in the following guise:

Theroem 1.5.4 (Induction principle): If a statement $\Phi(n)$ that depends on a natural number n is true for $n=0$ and if the implication $\Phi(n)\Rightarrow\Phi(S(n))$ is true for all n in $\mathbb{N}$ then $\Phi(n)$ is true for all n in $\mathbb{N}$.

The proof is trivial. Just translate all back to $\widehat{\mathbb{N}}$ with the help of C^{-1}. Take $A \underset{def.}{=} \left\{x\in\widehat{\mathbb{N}} \,\middle|\, \Phi\left(C^{-1}(x)\right)\right\}$. The conditions "$\Phi(0)$ is true" and $\forall(n\in\mathbb{N}):\Phi(n)\Rightarrow\Phi(S(n))$ mean that A is inductive. Then apply theorem 1.5.2 and finally go back to the numbers.

This theorem provides a very powerful tool for constructing proofs of theorems that have the form $\forall(n\in\mathbb{N}):\Phi(n)$.

The following list of theorems describes basic properties of the successor operation.

The proofs are simple exercises:

Theorem 1.5.5: $\forall(n\in\mathbb{N})\forall(m\in\mathbb{N}):\left(n\neq m \quad\Rightarrow\quad S(n)\neq S(m)\right)$

Theorem 1.5.6: $\forall(n\in\mathbb{N}):S(n)\neq 0$

Theorem 1.5.6: $\forall\left(n\in\mathbb{N}\setminus\{0\}\right)\exists(m\in\mathbb{N}):\quad n=S(m)$

We may joint the three theorems in a single formula: $S:\mathbb{N}\leftrightarrow\mathbb{N}\setminus\{0\}$.

The natural numbers can be ordered. One defines:

$$\forall(n\in\mathbb{N})\forall(m\in\mathbb{N}):\left(n<m \quad\underset{Def.}{\Leftrightarrow}\quad \widehat{n}\in\widehat{m}\right) \tag{1.5.11}$$

and

$$\forall(n\in\mathbb{N})\forall(m\in\mathbb{N}):\left(n\leq m \quad\underset{Def.}{\Leftrightarrow}\quad (n<m\vee n=m)\right) \tag{1.5.12}$$

The inverse relations are also written as $>$ and $\geq$ respectively. The relations $\leq$ and $\geq$ are total order relations.

The induction principle is an elegant way of talking about all natural numbers. This technique can also be used to define functions on $\mathbb{N}$. The idea is to specify the value $F(0)$ and to tell with a rule how the values $F(S(n))$ can be obtained starting from the values $F(n)$. Then this rule can determine the value $F(1)$ from the value $F(0)$. Next the same rule permits to determine the value $F(2)$ from the value $F(1)$ and so on. This sort of procedure is called a *recursive definition of a function*. Again this “and so on” has to be defined properly. This is done with the following theorem:

Theorem 1.5.7 (Recursion Theorem): Let A be a non empty set, $a \in A$ a given element, and $G : A \to A$ a given function. Then there exists a unique function $F : \mathbb{N} \to A$ such that $F(0) = a$ and $\forall(n \in \mathbb{N}):\ F(S(n)) = G(F(n))$.

The proof is not difficult but a bit lengthy. We shall introduce some abbreviations and show an auxiliary theorem. In mathematical literature such auxiliary theorems are called lemmas. We define the *strict and enclosing tail of a number*:

$$\text{for } n \in \mathbb{N} \text{ define} \qquad n_< \underset{def.}{=} \{m \in \mathbb{N} \mid m < n\} \tag{1.5.13}$$

$$\text{for } n \in \mathbb{N} \text{ define} \qquad n_\leq \underset{def.}{=} \{m \in \mathbb{N} \mid m \leq n\} \tag{1.5.14}$$

Lemma 1.5.7: Let A be a non empty set, $a \in A$ a given element, and $G : A \to A$ a given function. For any n in $\mathbb{N}$ there exists a unique function $F_n : n_\leq \to A$ such that $F_n(0) = a$ and $\forall(k \in n_<):\ F_n(S(k)) = G(F_n(k))$.

Proof of Lemma 1.5.7: We shall prove this lemma by induction. For $n = 0$ we have $0_\leq = \{0\}$, $0_< = \varnothing$ and the function with the only value $F_0(0) = a$ satisfies the conditions and is the only one that does that. Now let n in $\mathbb{N}$ be given and suppose that there is a unique function $F_n : n_\leq \to A$ such that $F_n(0) = a$ and $\forall(k \in n_<):\ F_n(S(k)) = G(F_n(k))$. We have to show the existence of a unique function $F_{S(n)} : (S(n))_\leq \to A$ such that $F_{S(n)}(0) = a$ and $\forall\left(k \in (S(n))_<\right):\ F_{S(n)}(S(k)) = G\left(F_{S(n)}(k)\right)$. We chose

$$F_{S(n)}(k) \underset{def.}{=} \begin{cases} F_n(k) & \text{if} \quad k \leq n \\ G(F_n(n)) & \text{if} \quad k = S(n) \end{cases} \tag{1.5.15}$$

This notation with a curly bracket enclosing several lines is common to define a function by pieces. One could equally well write several equations. This function satisfies the conditions $F_{S(n)}(0) = a$ and $\forall\left(k \in (S(n))_<\right):\ F_{S(n)}(S(k)) = G\left(F_{S(n)}(k)\right)$. It remains to show that this function is unique. If there were a second function $\tilde{F}_{S(n)}$ that satisfies these conditions this one would have to coincide with $F_{S(n)}$ on the set $n_\leq$ because of the fact that F_n was unique. It remains to show that $\tilde{F}_{S(n)}(S(n)) = F_{S(n)}(S(n))$. The expressions $\tilde{F}_{S(n)}(S(n))$ and $F_{S(n)}(S(n))$ have to fulfill the conditions $\tilde{F}_{S(n)}(S(n)) = G(F_n(n))$ and $F_{S(n)}(S(n)) = G(F_n(n))$ Then they have to be equal. This completes the proof.

The functions F_n of the lemma can be used to prove the theorem 1.5.7. The function whose existence is claimed is given by

$$\forall(n \in \mathbb{N}):\left(F(n) \underset{def.}{=} F_n(n)\right) \tag{1.5.16}$$

This function obviously satisfies the conditions of the theorem and its uniqueness follows from the uniqueness of the functions F_n and theorem 1.5.7 has been proven.

The recursion theorem can be used to define the sum and the multiplication of numbers. First one defines a function H_m for any m in $\mathbb{N}$ by means of the following recursion:

$$H_m(0) \underset{def.}{=} m \tag{1.5.17}$$

and

$$\forall(n\in\mathbb{N}):\Big(H_m(S(n)) \underset{def.}{=} S(H_m(n))\Big) \tag{1.5.18}$$

The values $F_m(n)$ depend on both variables; m and n. This defines a function $\Sigma:\mathbb{N}\times\mathbb{N}\to\mathbb{N}$ such that for all n in $\mathbb{N}$ and m in $\mathbb{N}$

$$\Sigma(\langle m,n\rangle) \underset{def.}{=} H_m(n) \tag{1.5.19}$$

Perhaps the reader might not have realized that this is simply the sum of the numbers m and n. The function Σ is usually written with the sign "+" and instead of writing $+(\langle m,n\rangle)$ one writes $m+n$. The successor operation can now be written in a more familiar form: $S(n)=n+1$ and we may terminate the use of the symbol S as successor function.

The reader can spend quite a time to prove the well known properties of the sum with nice inductive proofs:

Commutativity: $$\forall(n\in\mathbb{N})\forall(m\in\mathbb{N}):\quad m+n=n+m \tag{1.5.20}$$

Associativity: $$\forall(n\in\mathbb{N})\forall(m\in\mathbb{N})\forall(k\in\mathbb{N}):\quad m+(n+k)=(m+n)+k \tag{1.5.21}$$

Because of the associative property one may write simply $m+n+k$.

There is another and very natural way of defining the sum of numbers. For given numbers m in $\mathbb{N}$ and n in $\mathbb{N}$ let $\frown{m}$ and $\frown{n}$ be their respective standard representatives. Let a be an arbitrary object. For instance we may take $a=\varnothing$. Then we can form the set $\frown{n}\times\{a\}$. This set has the same cardinality as $\frown{n}$ and it consists of ordered pairs and therefore it has no common elements with $\frown{m}$. Then define

$$H_m(n) \underset{def.}{=} \left|\frown{m}\cup\left(\frown{n}\times\{a\}\right)\right| \tag{1.5.22}$$

This function satisfies the recursion (1.5.17), (1.5.18). Here the sum of natural numbers is very intuitively defined by joining sets without common elements and counting the number of resulting objects. This is exactly what a child does to build the notion of sum. The Cartesian product $\frown{n}\times\{a\}$ is only a simple trick to guarantee that the two sets have no common elements.

The multiplication can also be defined with a recursion. For any m in $\mathbb{N}$ we define a function $M_m:\mathbb{N}\to\mathbb{N}$ such that:

$$M_m(0) \underset{def.}{=} 0 \tag{1.5.23}$$

$$\forall(n\in\mathbb{N}):\Big(M_m(n+1) \underset{def.}{=} M_m(n)+m\Big) \tag{1.5.24}$$

Now the totality of these functions defines a mapping $\Pi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that

$$\Pi\left(\langle m,n \rangle\right) \underset{def.}{=} H_m(n) \tag{1.5.25}$$

This is the product of m and n. Similar to the case of the sum, one prefers to write an operational symbol between the numbers rather than a function symbol that takes the ordered pair as an argument. So instead of $\Pi\left(\langle m,n \rangle\right)$ one writes $m_\times n$.

In fact scientists are lazy and write simply mn without any symbol that indicates multiplication. This lazy way of writing a product has several disadvantages: 1) It is not a uniform way of writing a product because if both numbers are represented by constant names an ambiguity would arise; is 53 the product of 5 and 3 or is it the decimal representation of fifty-three? So in the case of two constant names one has to use a multiplication sing. 2) A similar ambiguity would arise if one admits the use of variables with several letters. So *mn* could simply be a name of a variable. The lazy way of writing products impedes the use of variables with several letters. 3) Who ever taught freshmen physics or mathematics courses will have noticed that many students misunderstand expressions like $f(x+\varepsilon)$. Instead of taking this as the value of the function f at the point $x+\varepsilon$ they interpret $f(x+\varepsilon)$ as a product of f and $x+\varepsilon$. Well, with some experience one can guess from the context what is meant. But scientific language should not work this way! The confounded students are completely right. We shall afford the luxury to write the multiplication sign. Fortunately in physics and other quantitative sciences multiplication of numbers is seldom. In most cases one multiplies values of physical quantities or other mathematical objects, and in some of theses cases we shall adhere to the lazy way of writing. For numbers we shall use the lazy notation only in well specified exceptional cases such as 2π and 4π.

Again we invite the reader so spend some time to prove the well known rules of multiplication:

Commutativity: $$\forall(n \in \mathbb{N})\forall(m \in \mathbb{N}): \quad m_\times n = n_\times m \tag{1.5.26}$$

Associativity: $$\forall(n \in \mathbb{N})\forall(m \in \mathbb{N})\forall(k \in \mathbb{N}): \quad m_\times(n_\times k) = (m_\times n)_\times k \tag{1.5.27}$$

Distributivity: $$\forall(n \in \mathbb{N})\forall(m \in \mathbb{N})\forall(k \in \mathbb{N}): \quad m_\times(n+k) = m_\times n + m_\times k \tag{1.5.28}$$

There is an alternative way of defining the product of natural numbers: For given natural numbers m and n let $\frown{m}$ and $\frown{n}$ be their respective standard representatives. We may define

$$m_\times n \underset{Def.}{=} \left|\frown{m} \times \frown{n}\right| \tag{1.5.29}$$

Sum und multiplication are functions of the type $F : A \times B \to C$. Such type of function is also called a *binary operation*. So + and $_\times$ are binary operations.

Now we have the basic mathematical elements that will be helpful to define what a physical quantity is. Other tools may be introduced on the way and some mathematical items shall be worked out in the exercises at the end of this section. Let us briefly summarize the main points: We have introduced basic logic notation, sets, relations, and functions. Especially the equivalence relations will play a fundamental role when we define quantities. Natural numbers have been defined and we have seen that the sum of numbers is a mapping of the kind $\Sigma : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ with certain properties.

Exercises:

E1.5.1: For any set A the Cartesian product $\varnothing \times A$ is empty. Therefore there exists only one element in $\mathsf{P}(\varnothing \times A)$, which is the empty set. Show that this relation is a function.

E1.5.2: For sets A and B one defines:

$$^{A}B \underset{Def.}{=} \left\{ x \in \mathsf{P}(A \times B) \middle| \; x : A \to B \right\} \tag{1.5.30}$$

With the help of this set of mappings one may define a binary operation on the set of natural numbers as follows

$$\forall (n \in \mathbb{N}) \forall (m \in \mathbb{N}) : \left(n^{m} \underset{Def.}{=} \left| {}^{\widehat{m}}\widehat{n} \right| \right) \tag{1.5.31}$$

Give an alternative recursive definition of this operation! Also show that this operation has the following properties:

$$\forall (k \in \mathbb{N}) \forall (l \in \mathbb{N}) \forall (m \in \mathbb{N}) : \quad k^{l} \times k^{m} = k^{(l+m)} \tag{1.5.32}$$

$$\forall (k \in \mathbb{N}) \forall (l \in \mathbb{N}) \forall (m \in \mathbb{N}) : \quad k^{m} \times l^{m} = (k \times l)^{m} \tag{1.5.33}$$

$$\forall (k \in \mathbb{N}) \forall (l \in \mathbb{N}) \forall (m \in \mathbb{N}) : \quad \left(k^{m}\right)^{l} = k^{m \times l} \tag{1.5.34}$$

E1.5.3: Let A be a set on which one has a commutative and associative binary operation "+". Let *n* and *m* be natural numbers and suppose $n < m$. Suppose one has a function $f : (m_{\leq} \setminus n_{<}) \to A$, where $m_{\leq}$ and $n_{<}$ are defined by (1.5.14) and (1.5.13). One defines the symbol $\sum_{k=n}^{m} f(k)$ to be the sum of the values of *f*. That means the sum $f(n) + f(n+1) + \dots\dots + f(m)$. The "…." is an indication of the type "and so on", which should be defined properly. Give a correct definition! Comment: The variable *k* appears in the expression $\sum_{k=n}^{m} f(k)$ in bound form. So it is a dummy variable. The value of the sum does not change if one writes a different variable name.

E1.5.4: Prove the following theorem inductively:

$$\forall (n \in \mathbb{N}) : \quad 2 \times \sum_{k=0}^{n} k = n \times (n+1) \tag{1.5.35}$$

E1.5.5: For *n* in $\mathbb{N}$ define

$$n! \underset{Def.}{=} \left| \left\{ x \in {}^{\widehat{n}}\widehat{n} \middle| \; x : \widehat{n} \leftrightarrow \widehat{n} \right\} \right| \tag{1.5.36}$$

Give an alternative recursive definition of $n!$ and show that your definition coincides with (1.5.36) for all n.