1.6 Quantities

The Joint Committee for Guides in Metrology (JCGM)[1] gives the following definition of quantity: "property of a phenomenon, body, or substance, where the property has a magnitude that can be expressed as a number and a reference". In this document the notion "magnitude", is explicitly declared an undefined one. This definition does not give any information about the requirements that the rules of comparison of a magnitude with the reference have to fulfill; the term "can be expressed as a number and a reference" is vague. What the JCGM-authors mean by reference, is a unit. But the notion of unit is equally undefined in that document. Finally this definition limits the notion of quantity to one dimensional ones. So velocity and force would not be considered physical quantities!

The JCGM definition of quantity is not able to elevate the intuitive notion of quantity to a conscious level. Let us see if we can do better than that! We shall not attempt to formulate a concise definition. The notion of quantity shall be explained gradually.

Quantities are properties of objects. But not every property of objects is a quantity! For instance beauty is not a quantity[2]. In order to call a property Q a quantity its definition should comply with certain requirements:

(a) The set of objects for which the quantity makes sense has to be defined. This set shall be called the ***domain of the quantity***, and we shall write it as D_Q.

(b) An equivalence relation $\sim_Q$ has to be defined on D_Q. The equivalent objects are said to ***have the same value of*** Q. A value of the quantity shall be defined as a mental object that characterizes the equivalence classes.

(c) A commutative and associative binary operation on the set of values has to be defined. This operation is called the sum of values.

(d) Some quantities permit also a comparison of values in the sense of smaller and larger.

These items need further explanation.

Comment concerning requirement (a): In general the *domain*, i.e. the set D_Q of admissible objects, is so diversified that it seems difficult to give a general definition. For instance, if you think of spatial distance we may talk of the distance of two atoms in a molecule, of the distance of two galaxies or of the distance of a car and a bicycle. In order to start with well defined sets one may first take some very restricted set of objects and later extend the quantity at hand to larger domains. For instance, in the case of spatial distance d we may begin with a domain D_d formed of pairs of points marked on some solid body.

The domain may consist of objects that can directly or indirectly interact with our senses. In this case we may call the quantity a *physical quantity*. But the objects could

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[1] Joint Committee for Guides in Metrology (JCGM), *International Vocabulary of Metrology, Basic and General Concepts and Associated Terms* (*VIM*), III ed., Pavillon de Breteuil : JCGM 200:2008, 1.1

[2] It is important to realize that non-quantitative properties exist and are relevant. The conspicuous success of quantitative sciences has seduced many people to employ quantities in inadequate places. For instance, it is inadequate to measure the relevance of intellectual work by the number of citations. Generally the imbeciles are the majority!

also be mental ones. For instance the cardinality of sets may be considered a quantity related to mental objects. According to theorem 1.3.1 there is no set of all sets. So if one adopts the requirement (a) in the case of the quantity *cardinality* one also has to start with some restricted domain which can be enlarged according to the necessities. For finite cardinality we may start with the initial domain $\widehat{\mathbf{N}}$ and later we may add objects like $\{\text{Danube, Tiber, Thames}\}$, $\{\text{J. S. Bach}\}$ etc..

Comment concerning requirement (b): The notion of equivalence relation has been discussed sufficiently in the section 1.4. What matters here is the case of physical quantities. These can be classified as primary and secondary quantities. The secondary ones are defined in terms of other quantities and in these cases the equivalence relation is inherited from the underlying quantities. In the case of a primary quantity the equivalence relation corresponds to some experimental procedure that compares two objects from the domain of the quantity and comes to a decision whether the objects are equivalent or not. Every single primary quantity has got its specific operational rule. At this point it is convenient to mention specific examples:

Example *spatial distance*: As mentioned above, the initial domain of spatial distance may be a set of pairs of marked points on a solid body. The equivalence relation can be defined as follows: the pair of points $\langle A,B\rangle$ is equivalent to the pair $\langle A',B'\rangle$ if and only if the spikes of a compass that fit into the points $\langle A,B\rangle$ also fit into the points $\langle A',B'\rangle$. If this is the case we shall write $d_{\langle A,B\rangle}=d_{\langle A',B'\rangle}$ and say "the pair of points $\langle A,B\rangle$ has the same distance as the pair of points $\langle A',B'\rangle$".

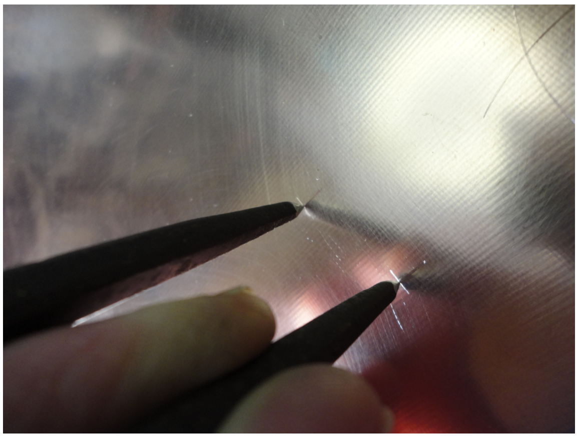
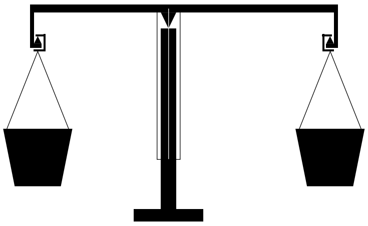


Fig.1.6.1 Spikes of a compass fitted to a pair of scratch marks on a metal surface.

Fig. 1.6.2 Compass used in a machine shop.

Example mass: The initial domain of this quantity may be some set of small material bodies that we find in our environment. In order to verify whether two bodies $\mathcal{A}$ and $\mathcal{B}$ are mass-equivalent, or we may say, have the same mass, we put them on the plates of a symmetric balance like the one of figure 1.6.3 and verify whether the balance remains in a horizontal equilibrium. If this is the case we shall write $m_{\mathcal{A}}=m_{\mathcal{B}}$. Further any of the material bodies is defined to be mass-equivalent to itself.

Fig. 1.6.3 Symmetric balance.



We have learned about abstraction. This knowledge can now be used to define the values of primary physical quantities:

A value of a quantity can be defined in several different ways, and generally the most appropriate way depends on the quantity. One possibility is to take the equivalence classes as values. Another one is given by the following definition: Let Q be a primary physical quantity with domain D_Q and operational equivalence relation $\sim_Q$. The *value* Q_a of an object a is any of the objects b in D_Q that satisfies $a \sim_Q b$ together with a mark Q that tells us that this new mental object is no longer an element of D_Q and the equality of theses new objects is defined as follows:

$$\forall\left(a \in D_Q\right)\forall\left(b \in D_Q\right):\left(Q_a = Q_b \underset{Def.}{\Leftrightarrow} a \sim_Q b\right) \tag{1.6.1}$$

Occasionally it can also be advantageous to use numbers as values. Later we shall see examples of quantities that are especially appropriated for defining their values as numbers. In the case of secondary quantities still other object can be used as values. What matters is that the chosen object can characterize the equivalence classes in a well defined manner.

The set of possible values shall be written as V_Q. Apart from the values that characterize equivalence classes of the domain the set V_Q may contain additional elements to guaranty the consistency of simple mathematical structures on V_Q. Generally these fictitious values turn out to have some practical application when one uses the quantity with some extension of the original domain.

The case of secondary quantities shall be discussed later when we study multiplication of quantities.

Comment concerning requirement (c): The sum rule of item (c) should allow the determination of a c in D_Q for any given a and b in D_Q so that c represents the value $Q_a + Q_b$. But this rule has to comply with certain conditions. First of all it has to be compatible with the equivalence classes. That means the value $Q_a + Q_b$ must not depend on the particular class representatives a and b. If a' and b' are objects such that $Q_a = Q_{a'}$ and $Q_b = Q_{b'}$ then the object c' that is obtained from a' and b' should have the same value as the object c that one obtains starting with a and b.

$$Q_a = Q_{a'} \quad \wedge \quad Q_b = Q_{b'} \quad \Rightarrow \quad Q_a + Q_b = Q_{a'} + Q_{b'} \tag{1.6.2}$$

Further the rule should fulfill the conditions:

$$Q_a + Q_b = Q_b + Q_a \tag{1.6.3}$$

$$Q_a + \left(Q_b + Q_c\right) = \left(Q_a + Q_b\right) + Q_c \tag{1.6.4}$$

In the case of primary physical quantities the sum of values is defined by means of a specific operational experimental procedure that is part of the definition of the quantity.

For instance in the case of mass we may define that a body c has the mass $m_a + m_b$ if the balance remains in horizontal equilibrium if we put c on one plate of the balance and the bodies a and b together on the other plate. The example of spatial distance will be treated later.

In the case of secondary quantities the sum is related to the sum of values of the underlying quantities and these cases shall be discussed in a different section.

This terminates our definition of quantity. In contrast to the JCGM definition we define what a *value* is. The undefined notion of *magnitude* will turn out to be a special case of *value*. Our definition mentions neither numbers nor *references* (units). Now we have to work out the details and see how numbers and units are related to values of quantities.

One can define the multiplication of numbers and values of quantities. Let Q be a quantity and let V_Q be the set of values that this quantity assumes on its domain. If n is some natural number larger than 0 one can define the multiple of values as a repeated sum.

$$\forall\left(v \in V_Q\right)\forall\left(n \in \mathbb{N}\right): \quad n\,v \underset{Def.}{=} \underbrace{v + \ldots\ldots + v}_{n \text{ times}} \tag{1.6.5}$$

In the case of multiplication of numbers and values of quantities we shall adhere to the lazy way of writing products, unless the value itself is also a number. Usually the number is written on the left hand side of the value, but we may also permit writing it on the right hand side:

$$\forall\left(v \in V_Q\right)\forall\left(n \in \mathbb{N}\right): \quad v\,n \underset{Def.}{=} n\,v \tag{1.6.6}$$

One may formulate this multiplication more elegantly with the help of a recursion. In this case it is convenient to start the recursion at $n = 0$. Many quantities have a special value, which shall be written as 0, that does not alter values if it is added to any value;

$$\forall\left(v \in V_Q\right): \quad v + 0 = v \tag{1.6.7}$$

If such a zero-value exists it is necessarily unique. We use the same symbol “0” that is used as a constant name of $|\varnothing|$. In principle this is not correct, but it so happens that it will not lead to any contradiction to consider a zero-value of a quantity the same thing as the cardinality of the empty set. If the quantity at hand does not have a zero-value one can invent one. For instance in the case of mass we may add to the domain a mental object, for instance the set $\varnothing$, if we imaging to “put” this object together with a material body on the plate of a balance the equilibrium condition of the balance will not suffer any alteration and according to the condition (1.6.7) we may consider the mass of the object $\varnothing$ to be 0. But such a dubious story of putting mental objects on plates of balances is not necessary. One may simply add an invented new element to the set of values. This is a first example of an enlargement of the set of values for the sake of mathematical simplicity. With this zero-value we can formulate a recursion to define multiplication with natural numbers:

$$\forall\left(v \in V_Q\right): \quad 0\,v \underset{Def.}{=} 0 \tag{1.6.8}$$

$$\forall\left(n \in \mathbb{N}\right)\forall\left(v \in V_Q\right): \quad (n+1)\,v \underset{Def.}{=} n\,v + v \tag{1.6.9}$$

With inductive proofs one easily shows the following properties of this multiplication:

$$\forall(n\in\mathbb{N})\forall(m\in\mathbb{N})\forall\left(v\in V_Q\right):\quad nv+mv=(n+m)v \tag{1.6.10}$$

$$\forall(n\in\mathbb{N})\forall\left(v\in V_Q\right)\forall\left(w\in V_Q\right):\quad nv+nw=n(v+w) \tag{1.6.11}$$

$$\forall\left(v\in V_Q\right):\quad 1v \;=\; v \tag{1.6.12}$$

$$\forall(n\in\mathbb{N})\forall(m\in\mathbb{N})\forall\left(v\in V_Q\right):\quad n{\times}mv=n(mv) \tag{1.6.13}$$

We used only one symbol of summation. Note however that the formula (1.6.10) contains two different types of sum. On the left hand side the +sign corresponds to an operational rule that associates a new value with the two values nv and mv, whereas on the right hand side the +sign indicates a simple sum of numbers. Despite this fact, the formula (1.6.10) does not define the sum of values in terms of sums of numbers! The operational rule of sum is also contained in the right hand side because the multiplication of numbers and values is defined in terms of sums of values.

The majority[3] of physical quantities have the following property: If $n\neq 0$ one can conclude from an equality $nv=nw$ that $v=w$. Further, the vast majority of quantities satisfies the condition that the sum of values defines an injective function $f_w:V_Q\to V_Q$, $f_w(v)=w+v$ for any fixed value w. A quantity that satisfies these conditions shall be called a ***linear quantity***.

$$Q \text{ is linear } \underset{Def.}{\Leftrightarrow}$$

$$\begin{aligned}&(a)\quad \forall\left(v\in V_Q\right)\forall\left(w\in V_Q\right)\forall(n\in\mathbb{N}):\left((n\neq 0\wedge nv=nw)\Rightarrow v=w\right)\\&\wedge\\&(b)\quad \forall\left(w\in V_Q\right):f_w:V_Q\to V_Q \text{ with } f_w(v)=w+v \text{ is injective.}\end{aligned} \tag{1.6.14}$$

For any value v of a linear quantity Q and any natural number n and non-zero natural number k there is at most one value w such that $nv=kw$. If this value always exists then we shall call the quantity a ***continuous linear quantity***. So for continuous linear quantities one has:

$$\forall\left(v\in V_Q\right)\forall(n\in\mathbb{N})\forall(k\in\mathbb{N}):\left(k\neq 0\Rightarrow\exists!\left(w\in V_Q\right):nv=kw\right) \tag{1.6.15}$$

This remarkable property motivates to generalize the notion of number. One invents a new type of number such that the uniquely determined value w can be written as a multiple of the value v :

$$w=qv \tag{1.6.16}$$

where q is a new kind of number, which is determined by the numbers n and k. Again, we shall allow to write the number on either side of the values; $vq \underset{Def.}{=} qv$. The definition of these new objects should exactly take the properties into account that are needed for the application (1.6.16). A pair of natural numbers n and k with $k\neq 0$ determines the new object q, but two pairs that differ only by a common multiplier will determine the same value w. For any natural number m with $m\neq 0$ the pairs

[3] There exist exceptions. For instance the quantity *angle* may be defined in several different ways. One way gives a circular range of values and the implication $nQ_A=nQ_C \;\Rightarrow\; Q_A=Q_C$ does not hold for $n\neq 0$. Another exception is the pseudo momentum of electrons in a crystal lattice.

$\langle n,k\rangle$ and $\langle m\times n,m\times k\rangle$ will establish the same relation between values v and w: For all n, for all $k\neq 0$, $m\neq 0$ and all values v and w one has $nv=kw \Leftrightarrow m\times nv=m\times kw$. That looks like an equivalence relation defined on the set $\mathbb{N}\times(\mathbb{N}\setminus\{0\})$. We shall abbreviate the set of positive natural numbers $\mathbb{N}\setminus\{0\}$ and write $\mathbb{N}_1$. Then this equivalence relation can be formulated as follows:

$$\forall(\langle n,k\rangle\in\mathbb{N}\times\mathbb{N}_1)\forall(\langle m,r\rangle\in\mathbb{N}\times\mathbb{N}_1):\left(\langle n,k\rangle\underset{/}{\sim}\langle m,r\rangle \underset{def.}{=} n\times r=m\times k\right) \quad (1.6.17)$$

The new kind of number is defined by an abstraction process associated with this equivalence relation. Mathematicians usually perform this abstraction process in a very simplified way. They define the new object q simply as an equivalence class generated by the equivalence relation. For any pair $\langle n,k\rangle\in\mathbb{N}\times(\mathbb{N}_1)$ they write the equivalence class that contains this pair as $[\langle n,k\rangle]_{\underset{/}{\sim}}$:

$$[\langle n,k\rangle]_{\underset{/}{\sim}} \underset{def.}{=} \left\{x\in\mathbb{N}\times\mathbb{N}_1 \,\middle|\, x\underset{/}{\sim}\langle n,k\rangle\right\} \quad (1.6.18)$$

and they treat these classes as the new numbers. The set of these new numbers is called the set of *non-negative rational numbers* and it shall be written as $\mathbb{Q}_{\geq 0}$. The following rules of calculus are defined:

$$[\langle n,k\rangle]_{\underset{/}{\sim}}+[\langle m,r\rangle]_{\underset{/}{\sim}} \underset{def.}{=} [\langle n\times r+m\times k,k\times r\rangle]_{\underset{/}{\sim}} \quad (1.6.19)$$

$$[\langle n,k\rangle]_{\underset{/}{\sim}}\times[\langle m,r\rangle]_{\underset{/}{\sim}} \underset{def.}{=} [\langle n\times m,k\times r\rangle]_{\underset{/}{\sim}} \quad (1.6.20)$$

One can verify that the sum and multiplication defined this way obey the usual rules. There is a natural embedding of the natural numbers into the set of these new numbers. This embedding is given by $E(n)\underset{def.}{=}[\langle n,1\rangle]_{\underset{/}{\sim}}$.

At first sight the idea to take the equivalence classes as the new mental objects seems to be more elegant than our clumsy way of abstracting: "The new object is any of the equivalent objects together with a mark that tells us that this new object is no longer ….". But this clumsy way of abstraction has an advantage:

The classes are sets and therefore the equality is defined by the extensionality axiom. We do not have the right to change that. This obliges us to use the embedding function E. So, the natural numbers are not contained in the set of rational numbers! To write $n=[\langle n,1\rangle]_{\underset{/}{\sim}}$ is formally incorrect and therefore it is equally incorrect to write $2\in\mathbb{Q}_{\geq 0}$ or $\mathbb{N}\subset\mathbb{Q}_{\geq 0}$. But the fact is that everybody writes the latter formulas and declares them as true ones. One might say, well let us do things the correct way, do not write $\mathbb{N}\subset\mathbb{Q}_{\geq 0}$ and use the embedding mapping! But this decision is quite impractical because there come still more embeddings: the introduction of negative numbers requires another embedding and the introduction of irrational numbers still another one. The simple task to write the number *two* as a real number would occupy have a line! Our clumsy way of abstracting allows defining these new numbers in such a way that formulas such as $2\in\mathbb{Q}_{\geq 0}$ and $\mathbb{N}\subset\mathbb{Q}_{\geq 0}$ are true formulas and the embedding mapping is not necessary.

Our way of abstracting defines the abstract object as an entity that characterizes the equivalence classes. These entities are obtained from the elements of the classes by transforming these elements into new objects and defining the equality of the new objects. In the present case we shall only transform some of the elements into new objects by means of an associated mark. First we shall enlarge the set of objects. Instead of starting with the set $\mathbb{N}\times\mathbb{N}_1$ we start the construction of rational non-negative numbers from $\left(\mathbb{N}\times\mathbb{N}_1\right)\cup\mathbb{N}$. Then we extend the equivalence relation $\underset{/}{\sim}$ to that larger set with the following defining formulas:

$$\forall\left(n\in\mathbb{N}\right)\forall\left(m\in\mathbb{N}\right):\left(n\underset{/}{\sim}m\quad\underset{def.}{\Leftrightarrow}\quad n=m\right)\tag{1.6.21}$$

$$\forall\left(n\in\mathbb{N}\right)\forall\left(\langle m,k\rangle\in\mathbb{N}\times\mathbb{N}_1\right):\left(n\underset{/}{\sim}\langle m,k\rangle\quad\underset{def.}{\Leftrightarrow}\quad\langle n,1\rangle\underset{/}{\sim}\langle m,k\rangle\right)\tag{1.6.22}$$

The natural numbers are declared to be rational numbers right away without any additional mark. From the pair $\langle n,k\rangle$ we form a new object with the help of a special mark, written as "/" or as "$-$" that tells us that this new object is now longer an ordered pair. The elements of the pair are written together with the mark in the form n/k or $\frac{n}{k}$. The equality of these objects is induced by the equivalence relation given by (1.6.17), (1.6.21), and (1.6.22). Instead of the rules (1.6.19) and (1.6.20) one has

$$\frac{n}{k}+\frac{m}{r}=\frac{n\times r+m\times k}{k\times r}\tag{1.6.23}$$

$$\frac{n}{k}\times\frac{m}{r}=\frac{n\times m}{k\times r}\tag{1.6.24}$$

With these new numbers one may write the value w of formula (1.6.15) as

$$w=\frac{n}{k}v\tag{1.6.25}$$

As one can see, the JCGM definition does not at all reveal how human quantitative cognition works. The properties of quantities motivate the very construction of rational numbers. Therefore one should not define quantities with the help of numbers. The notion of number in its general form is a consequence of the existence of quantities.

Of course one could also introduce the rational numbers without this motivation and use purely formal arguments instead. This goes as follows: Let n, m and x be variables that describe natural numbers. For any given values of n and m one might try to find a number x such that the equation $n\times x=m$ holds true. This type of *riddle* is very common in mathematics and usually it is called *equation*. This name is not very precise because $5+3=8$ could also be called an equation, and this latter one is not a riddle. We propose the following nomenclature; we shall call all formulas of the type expression1 = expression 2 *equalities* and when there is any kind of unknown object involved which shall be determined so that the equality is in fact a riddle we shall call this formula an *equation*. Now the equation $n\times x=m$ sometimes does not have solutions. With n = 2 and m = 6 a solution exists, but with n = 2 and m =7 there is no solution. Then one may take the decision: if there is no solution let us invent one; $x=7/2$.

In a similar way one may invent negative numbers in order to guarantee the existence of solutions of equations of the type $a+x=b$. Any pair of rational non-negative numbers a and b determines a number x. But if two pairs $\langle b,a\rangle$, $\langle d,c\rangle$ satisfy the relation $b+c=d+a$ they will determine the same number x. So we have again an equivalence relation and the new numbers are defined with the corresponding process of abstraction. The details are completely analogous to the previous case of division. Again one better avoids the apparently elegant way of abstracting with equivalence classes and uses our cumbersome method so that one has the right to consider the non-negative rational numbers a subset of the rational ones. The resulting set of rational numbers shall be written as $\mathbb{Q}$.

The rational numbers together with the binary operations "+" and " $\times$ " form a structure that mathematicians would call a *commutative field* [4]. These operations fulfill the following conditions:

(a) " $+$ " and " $\times$ " are commutative.

(b) " $+$ " and " $\times$ " are associative.

(c) There are neutral elements "0" and "1" such that
$\forall(a\in\mathbb{Q}):(a+0=a \quad \wedge \quad a\times 1=a)$

(d) The neutral elements are different: $0\neq 1$

(e) $\forall(a\in\mathbb{Q})\exists(b\in\mathbb{Q}):\quad a+b=0$

(f) $\forall(a\in\mathbb{Q}\setminus\{0\})\exists(b\in\mathbb{Q}):\quad a\times b=1$

(g) $\forall(a\in\mathbb{Q})\forall(b\in\mathbb{Q})\forall(c\in\mathbb{Q}):\quad a\times(b+c)=(a\times b)+(a\times c)$ (distributive law)

The element b whose existence is stated in the item (e) is unique and it is usually written as $-a$ and the element b of item (f) is also unique and it is written as a^{-1}. And one defines the binary operation " $-$ " as:

$$\forall(a\in\mathbb{Q})\forall(b\in\mathbb{Q}):\quad a-b \underset{Def.}{=} a+(-b) \tag{1.6.26}$$

Integer powers can be defined with the same type of recursive procedures that was used in the exercise E 1.5.2. Further one agrees upon the convention that multiplication "binds stronger" than the sum. That means that one may write the right hand side of the equation of item (f) without the parentheses. So one defines:

[4] Here we have encountered a notorious problem in science. Every scientific field abuses words from natural language. So, for instance, mathematicians use the good old *field* not to plant vegetables. Physicists use the same word for a certain type of physical systems. Physicists that work in *field theory* have to use a lot of mathematics. They calculate *cross sections,* which term for a mathematician has again a completely different meaning. It has come the time to arrange things in a proper way! One might invent completely new artificial and uniquely determined words. But it is evident that such an attempt would fail the same way as all attempts to propagate artificial languages, such as Esperanto, have failed. Our suggestion is to use prefixes whenever there is any risk of confusion. So the above mentioned mathematical structure can be called a *math*-field and the physical system a *phys*-field. In other cases one may also have *chem*- (chemical), *med*- (medical) , *biol*- (biological), *eng*-(engineering) , *finac*-(financial) etc. objects. In a scientific text a word with its ordinary meaning, which could be confused with a technical term, shall be written with a prefix "*com.*-" (common). So for example: In the *com.*-field of mathematics a field is a structure.

$$a \times b + c \times d \underset{Def.}{=} (a \times b) + (c \times d) \tag{1.6.27}$$

Apart from theses algebraic properties of the rational numbers they also have an order relation. The relations <, >, $\leq$, and $\geq$ that were defined in $\mathbb{N}$ can be extended to $\mathbb{Q}$ and one has

$$\forall(a \in \mathbb{Q})\forall(b \in \mathbb{Q})\forall(c \in \mathbb{Q}): \quad a < b \Rightarrow a + c < b + c \tag{1.6.28}$$

$$\forall(a \in \mathbb{Q})\forall(b \in \mathbb{Q})\forall(c \in \mathbb{Q}): \quad (a < b \wedge 0 < c) \Rightarrow a \times c < b \times c \tag{1.6.29}$$

$$\forall(a \in \mathbb{Q})\exists(n \in \mathbb{N}): \quad a < n \tag{1.6.30}$$

We shall skip the demonstration of the properties (a) – (f) and (1.6.28) - (1.6.30) in order not to deviate too much from the issue at hand.

So let us come back to that issue: We have defined the multiplication of non-negative rational numbers and values of a continuous linear quantity. This multiplication satisfies formulas analogous to the rules (1.6.10) - (1.6.13):

$$\forall(r \in \mathbb{Q}_{\geq 0})\forall(s \in \mathbb{Q}_{\geq 0})\forall(v \in V_Q): \quad r\,v + s\,v = (r + s)\,v \tag{1.6.31}$$

$$\forall(r \in \mathbb{Q}_{\geq 0})\forall(v \in V_Q)\forall(w \in V_Q): \quad r\,v + r\,w = r(v + w) \tag{1.6.32}$$

$$\forall(v \in V_Q): \quad 1v = v \tag{1.6.33}$$

$$\forall(r \in \mathbb{Q}_{\geq 0})\forall(s \in \mathbb{Q}_{\geq 0})\forall(v \in V_Q): \quad r \times s\,v = r(s\,v) \tag{1.6.34}$$

We shall demonstrate only the first rule and leave the demonstration of the remaining ones as an exercise. Let $r = n/m$ and $s = k/l$ with n, k, from $\mathbb{N}$ and m, l from $\mathbb{N}_1$ and v from V_Q be given arbitrarily. The definition of the product of non-negative rational numbers and values gives:

$$n\,v = m(r\,v) \quad \text{and} \quad k\,v = l(s\,v) \tag{1.6.35}$$

With the rule (1.6.13) we conclude

$$l \times n\,v = l \times m(r\,v) \quad \text{and} \quad m \times k\,v = m \times l(s\,v) \tag{1.6.36}$$

We may add these equations and use the rules (1.6.10) and (1.6.11) to obtain

$$(l \times n + m \times k)\,v = l \times m((r\,v) + (s\,v)) \tag{1.6.37}$$

According to the definition of product of non negative rational numbers and values this means

$$\frac{l \times n + m \times k}{l \times m} v = r\,v + s\,v \tag{1.6.38}$$

and according to the definition of sum of rational numbers this gives the desired result $r\,v + s\,v = (r + s)\,v$.

Now one may ask whether one can multiply values of a continuous linear quantity and negative numbers. Let v be a value of a continuous linear quantity and $q > 0$ some positive rational number. If the equation

$$qv + x = 0 \tag{1.6.39}$$

has a solution we know that this solution is unique because of point (b) of the definition (1.6.14). Then we may define the product of $(-q)$ and v to be that solution. With this definition the rules (1.6.31) - (1.6.34) can be extended to negative numbers. If the equation (1.6.39) does not have a solution in the set of values V_Q one can invent new values simply by adding the expressions $(-q)v$ as new mental objects to the original set and by identifying objects appropriately so that the rules (1.6.31) - (1.6.34) are valid also for negative numbers.

At first sight such artificial enlargement of the set of values may appear to be a mere question of formal beauty of mathematical structure. But this is not the case. Generally these invented new values turn out to have practical applications. We may see this with the simple example of mass. In the domain of mass formed of small material bodies in our environment we cannot find any solutions of equations of the type (1.6.39). There are no bodies with a negative mass. But the invented negative mass values can be *com.*-real[5] values of a physical quantity if we make a *secondary* use of the quantity mass. Let A and B be two separated spatial regions that contain some material bodies. In a certain time interval there may happen a transport of bodies between these regions. Then we may define a physical quantity called "net mass transported from A to B". This quantity is the difference of the mass of bodies that have been transported from A to B and the mass of the bodies that have been transported from B to A. The net mass transported from A to B has values in the enlarged set of values V_m that includes the negative mass values.

With the inclusion of products of negative numbers and values, the values of a continuous linear quantity form another important mathematical structure; a *linear space*.

A linear space over a commutative field F with a sum "+" and a product " $\times$ " is a set V in which a commutative and associative sum "+" is defined such that

$$\exists(\mathrm{o}\in V)\,\forall(x\in V):\quad x+\mathrm{o}=x \tag{1.6.40}$$

$$\forall(x\in V)\,\exists(y\in V):\quad x+y=\mathrm{o} \tag{1.6.41}$$

together with a product of elements of F and V (which is usually written the lazy way without any sign) such that

$$\forall(r\in F)\,\forall(s\in F)\,\forall(v\in V):\quad r\,v+s\,v=(r+s)\,v \qquad (1.6.42)^6$$

$$\forall(r\in F)\,\forall(v\in V)\,\forall(w\in V):\quad r\,v+r\,w=r(v+w) \tag{1.6.43}$$

$$\forall(v\in V):\quad 1_F\,v \;=\; v \tag{1.6.44}$$

$$\forall(r\in F)\,\forall(s\in F)\,\forall(v\in V):\quad r\times s\,v=r(s\,v) \tag{1.6.45}$$

where 1_F is the neutral elements in F with respect to multiplication in F. So we have come to the important result:

[5] See footnote 4)

[6] As in the case of formula (1.6.31) we use the + sign for two types of sum.

The values of a linear quantity form a linear space over the field of rational numbers, or can be extended to a linear space over $\mathbb{Q}$. This space shall be called the *value space* of the quantity. The quantity establishes a mapping that maps the domain of the quantity into its value space.

In this statement we mentioned linear quantities without the requirement of continuity. In the case of a linear quantity that is not continuous the extension of the value space takes care of the missing values.

The combination of sum in a linear space V and multiplication of values from V and elements from the *math*-field F leads to expressions of the form $\alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_n v_n$, where $\alpha_1, \alpha_2, ...$ are elements of the field F and $v_1, v_2, ...$ are elements of V. Such type of expression is called a ***linear combination of the values*** $v_1, v_2, ...$

In the case of a linear quantity Q one may now try to find a set of basic values $\{b_1, ..., b_n\}$ in such a way that any value of Q can be written as a linear combination of these basic values: That means the basic values should be chosen in such a way that for all v in V_Q there exist rational numbers $\alpha_1, \alpha_2, ...$ such that

$$v = \alpha_1 b_1 + ... + \alpha_n b_n \qquad (1.6.46)$$

With the notation that has been introduced with the exercise E1.5.3 we may write the equality (1.6.46) in the elegant form

$$v = \sum_{k=1}^{n} \alpha_k b_k \qquad (1.6.47).$$

In this sentence we used the general word "equality", but very often a formula of the type (1.6.46) is in fact an equation. Imagine one has a concrete physical object and v is the value of the quantity Q that can be attributed to that object. For given basic values $b_1, ...b_n$ one can try to determine numbers $\alpha_1, \alpha_2, ...$ such that (1.6.47) holds true. Then (1.6.47) is in fact an equation and the unknowns are the numbers $\alpha_1, \alpha_2, ...$. Applying the operational rules of comparison of values and sum of values one may determine the numbers $\alpha_1, \alpha_2, ...$ experimentally. Such process of experimental determination of a solution of equation (1.6.47) is called a ***measurement of the value v.*** In general, we shall call any experimental realization of the quantity defining operational rules that results in a statement about values of the quantity **a measurement of a quantity *Q***.

Some textbooks of elementary physics define a physical quantity as a property that can be measured, but without defining what it means to measure something. A proper definition of quantity permits defining the notion of measurement[7].

[7] An early attempt to elaborate a theory of measurement is due to Helmholtz (Helmholtz, H.v.,. "Zählen und Messen erkenntnis-theoretisch betrachtet". *Philosophische Aufsätze Eduard Zeller gewidmet.* Leipzig, 1887.). Hölder elaborated further an axiomatic theory of measurement (Hölder, O. "Die Axiome der Quantität und die Lehre vom Mass". *Ver. Verh. Kgl. Sachisis. Ges. Wiss. Leipzig. Math-Phys. Classe* 53, 1-64 (1901)) see also Krantz, D.H., Duncan, R., Suppes, L.P., and Tversky, A.: Foundations of Measurement. Vol. 1-3. San Diego: Academic Press 1971. Narens, L., and Luce, R.D., "The algebra of measurement" *J. Pure and Applied Algebra* 8, 197-233 (1976). and Narens, L. *Abstract Measurement Theory* MIT Press Cambridge, Massachusetts London, England ISBN-10: 0262140373. (1985)

The minimal number of basic values necessary to write any arbitrary value of a quantity depends on the quantity. This minimal number shall be called the dimension of the quantity[8]. For example, mass and spatial distance are one-dimensional. Velocity, acceleration and force are three-dimensional. In relativistic physics we shall encounter many four-dimensional quantities. And there exist many higher dimensional quantities. Especially 6-dimensional quantities occur quite often.

In the case of one-dimensional quantities one calls the basic value a ***unit.*** Values of one-dimensional quantities may be called *magnitude* in order to attribute some sense to the JCGM-definition of quantity. Any non-zero value of a one-dimensional quantity may be chosen as a unit. But it is convenient to choose a value that can be known and reproduced experimentally by many people in order to facilitate communication. In the case of mass the official unit is called kilogram and in the case of distance it is called meter. How these units are determined shall be discussed much later.

We shall write the constant names of officially defined units with normal (non-italic) letters. So if x is a variable name of a spatial distance an equality $x = 5\,\mathrm{m}$ is a syntactically correct formula. Scientific literature frequently uses prefixes that symbolize numerical factors of the type 10^k were k is some integer. The official nomenclature is shown in the following table:

Table 1.6.1 Prefixes of units.

Factor	Prefix	Name	Factor	Prefix	Name
10^{24}	Y	Yotta	10^{-1}	d	deci
10^{21}	Z	Zetta	10^{-2}	c	cent
10^{18}	E	Exa	10^{-3}	m	milli
10^{15}	P	Peta	10^{-6}	μ	micro
10^{12}	T	Tera	10^{-9}	n	nano
10^{9}	G	Giga	10^{-12}	p	pico
10^{6}	M	Mega	10^{-15}	f	femto
10^{3}	k	Kilo	10^{-18}	a	atto
10^{2}	h	Hecto	10^{-21}	z	zepto
10^{1}	da	Deca	10^{-24}	y	yocto

[8] Usually physical literature uses the word dimension also with a different meaning. For instance one writes sentences like the following one: "The dimension of mass is different from the dimension of distance"). In this book we shall not use the word dimension this way. Our notion of *value space* can replace that. The cited sentence can be written as: "The value space of mass is different from the value space of distance".

With the exceptions of Kilo, Hecto, and Deca all prefixes with positive exponents are written with capital letters and all prefixes with negative exponents with lower case letters. The exception is generally justified saying that a prefix "K" could be confounded with the unit K (Kelvin). This argument is not at all convincing. In fact who teaches engineering courses of transport of heat and matter will have noticed that students in fact confound the constant name of meter with the prefix milli. Thus the official rules are poor. Bad rules should not be followed they should be changed! In this book we shall write prefixes with letters smaller than the units. So for example millimeter shall be written as mm so that there is no chance of interpreting a unit as a prefix. This allows eliminating the exceptions; all prefixes with positive exponents shall be written with capital letters. So for instance, a kilometer shall be written as Km and Kelvin meter as $\mathrm{K\,m}$.

It is important to note that the value of a quantity of a given object does not depend on the choice of unit. One says the value is *invariant*. If we change the chosen unit the number in front of the unit will change in such a way that the value remains the same. So, for instance, doubling the unite will reduce the number in front correspondingly by a factor ½. This behavior is called *contra-variant*.

It is a common mistake to believe that the sum of values of one-dimensional quantities can be defined expressing the values in terms of a unit. If we have a pair of points with distance 2m and another one with distance 5m the sum of these values is 7m. But this does not define the some of distance! It will not tell us which pairs of points have the distance 7m!. The sum of values has to be defined operationally and only when this definition exists one can talk of units. Remember that there are two kinds of sums evolved with linear spaces! This point is left completely unclear in the JCGM-definition of quantity.

We shall terminate this section with some experimental exercises, but before coming to that point we shall explain one very important mathematical issue that is excellently motivated by the preceding discussion. Let us come back to the equation (1.6.47) with a given value v and given basic values $b_1,....b_n$ so that the numbers $\alpha_1, \alpha_2,....$ are the unknowns of the equation. Concerning equations there are always two immediate questions: (1) are there any solutions? and (2) if so, is the solution unique?

In the case that the first question receives a positive answer for all values v one calls the set of basic values a ***basis of the space*** V_Q. Of course it would be very convenient if one could choose a basis such that the existing solution is also unique. This can be achieved with ***linearly independent sets.*** We shall discuss this very important notion in a general form with some linear space V and commutative field F. First we state a simple theorem:

Theorem 1.6.1 Let V be a linear space with a commutative field F. Let o be the neutral element in V and let 0_F and 1_F be the neutral elements with respect to sum and multiplication in F respectively. Then the following statements are true:

$$\mathrm{o} \text{ is unique} \tag{1.6.48}$$

$$\forall(a \in V)\exists!(x \in V): \quad a + x = \mathrm{o} \tag{1.6.49}$$

$$\forall(a \in V): \quad 0_F a = \mathrm{o} \tag{1.6.50}$$

$$\forall(\alpha \in F): \quad \alpha\,\mathrm{o} = \mathrm{o} \tag{1.6.51}$$

$$\forall(a \in V)\forall(\alpha \in F):\ \left(\alpha a = \mathrm{o} \ \Rightarrow\ \left(\alpha = 0_F \vee a = \mathrm{o}\right)\right) \tag{1.6.52}$$

We leave the demonstration as an exercise.

Let $\{b_1,....,b_k\}$ be a finite[9] subset of a linear space V over a field F. This set is said to be ***linearly independent*** if and only if the only solution of the equation $\sum_{r=1}^{k}\alpha_r b_r = \mathrm{o}$ is the trivial[10] solution i.e. $\alpha_1 = 0_F,\ \alpha_2 = 0_F, ...\ \alpha_k = 0_F$:

$$\{b_1,....,b_k\} \text{ is linearly independnet } \underset{Def.}{\Leftrightarrow}$$
$$\forall(\alpha_1 \in F)...\forall(\alpha_k \in F):\left(\sum_{r=1}^{k}\alpha_r b_r = \mathrm{o} \ \Rightarrow\ \forall\left(i \in \{1,..k\}\right):\ \alpha_i = 0_F\right) \tag{1.6.53}$$

If $\{b_1,....,b_k\}$ is a linearly independent set a solution of an equation $v = \sum_{k=1}^{n}\alpha_k b_k$ is necessarily unique: Suppose $\alpha_1, \alpha_2, ...$ and $\tilde{\alpha}_1, \tilde{\alpha}_2, ...$ are solutions. Then one has

$$\sum_{k=1}^{n}\alpha_k b_k = \sum_{k=1}^{n}\tilde{\alpha}_k b_k \tag{1.6.54}$$

or

$$\sum_{k=1}^{n}\left(\alpha_k - \tilde{\alpha}_k\right) b_k = \mathrm{o} \tag{1.6.55}$$

Then linear independence implies $\left(\alpha_k - \tilde{\alpha}_k\right) = 0_F$ for all k and consequently $a_k = \tilde{a}_k$ for all k.

One can construct a basis in the following way: One takes some non-zero value b_1 from V. Because of (1.6.52), the set $\{b_1\}$ is linearly independent. Next one successively subjoins other elements b_k such that the sets $\{b_1, b_2\}$, $\{b_1, b_2, b_3\}$,.... are all linearly independent. This process may stop at some point, when it is impossible to find any more elements that can be subjoined without destroying linear independence. So let us suppose we have come to that point forming a set $\{b_1,....,b_n\}$ with some number $n \geq 1$. Then this maximal set is necessarily a basis in V. To show that $\{b_1,....,b_n\}$ is a basis we have to show that $v = \sum_{k=1}^{n}\alpha_k b_k$ has a solution (the αs are the unknowns) for any v in V. So let v be given arbitrarily. The fact that $\{b_1,....,b_n\}$ was maximal implies that the set $\{b_1,....,b_n, v\}$ is not linearly independent. Then the equation

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[9] One can also consider infinite sets but for the time being we shall restrict the discussion to the finite case.

[10] The word "trivial" is used as a technical term and means extremely simple. The origin of this word is related to the number three. In antique and medieval times the university studies stared with three basic disciplines: Grammar (including literature), Dialectic (logic), and Rhetoric (which included law and ethics). This basic study was called Trivium. Supposedly this basic stuff is the simple one, which originated the meaning of trivial.

$\sum_{k=1}^{n} \beta_k b_k + \beta_{n+1} v = \mathrm{o}$ has a non-trivial solution. From the linear independence and (1.6.50) we conclude $\beta_{n+1} \neq 0_F$. Thus there exists an element γ in F such that $\gamma \times \beta_{n+1} = 1_F$. Then $\alpha_k = -\gamma \times \beta_k$ is a solution of the equation $v = \sum_{k=1}^{n} \alpha_k b_k$. This completes the proof. The number n is the dimension of the linear space. It may happen that the process of subjoining independent elements never ends. Then the linear space is said to have infinite dimensionality. Much later we shall see that infinite dimensional linear spaces are extremely important in physics, but the mathematical details of infinite dimensional spaces shall be explained when these spaces are needed.

In 2007 in the physics course of the Federal University of Juiz de Fora (Brazil) we discussed the notion of physical quantity in the very first laboratory freshman class. The equivalence relation, the sum rule, and the multiplication of values with rational numbers were exemplified with a balance shown in the photograph Fig. 1.6.4

Fig. 1.6.4 Symmetric balance used to exemplify the notion of physical quantity with the example mass.

The students received a set of small metal objects (shown in figure 1.6.5). They had to verify the properties of the equivalence relation and to classify the objects. Then they had to determine the masses of all objects using the mass of the nut as a unit. The objects had been fabricated so that their masses were rational fractions with small integers times the mass unit.

The students received the task to think about the question “what is a physical quantity” with expressive interest.

Exercise: Construct a balance of the type shown in figure 1.6.4 and fabricate 9 metallic bodies with mass values m_0, $\frac{2}{3}m_0$, and $\frac{3}{2}m_0$ (3 bodies for each value). Then elaborate a strategy to teach the contents of sessions 1.1-1.6 in an intuitive rather than a formal way.

Fig. 1.6.5 Metal objects used to exemplify the quantity mass.