1.8 Secondary Quantities.

In section 1.2 we saw that one can build new formulas from old ones with the help of the logical connectives $\neg$, $\vee$, $\wedge$, $\Rightarrow$, and $\Leftrightarrow$. In a similar way one may define new quantities from old ones.

Let us start with a very simple, almost trivial connection of old and new quantity. Let Q be some linear quantity and let α be some fixed number. Then we may define a quantity αQ such that the value of that quantity of any object A in the domain of Q has the value αQ_A .

Next one often finds objects that permit defining several quantities of the same kind with values in the same value space. For instance, think of rectangles. A rectangle permits defining two quantities of the type spatial distance; the width w and the height h. Now we can form a new quantity $s = 2w + 2h$, which is the length of the periphery.

These ways of forming new quantities from old ones are very simple because they use only the operations that are already defined in the value space. A much more interesting way of getting new quantities is multiplying old ones. This needs a specific definition. Multiplication of numbers may be introduced by repeated summing. So 3×5 can be defined as $5+5+5$. But this sort of definition does not work if one wants to multiply mass and acceleration.

Let Q and F be two linear and continuous quantities with finite dimensional value spaces V_Q and V_F respectively. In order to define a product of Q and F we have to assume that the domains of these quantities have a non-trivial intersection, otherwise it would not make much sense to combine Q and F to form a new quantity. If the original domains D_Q and D_F have an empty or a trivially small intersection one may enlarge them so that the new domains have a reasonable intersection that is large enough to form the domain of the new quantity $Q \otimes F$.

If A is an object from this domain it has values Q_A and F_A of the respective quantities Q and F. It is natural to require that theses values determine a unique value of the new quantity $Q \otimes F$. If A and B are objects such that $Q_A = Q_B$ and $F_A = F_B$ we demand that $(Q \otimes F)_A = (Q \otimes F)_B$.

But this legacy of equivalence is not the only source of equality of values. An ordered pair of values $\langle Q_A, F_A \rangle$ determines a value of the quantity $Q \otimes F$. But several different pairs may determine the same value of the product. The question which pairs should be taken to define the same value of the product depends on the notion of product itself. What kind of binary operations deserve the name of product? Well, anything deserves a special name only if that thing turns out to be useful or important. It so happens that binary operations that obey the same rules of multiplication of numbers usually have a lot of importance. If α, β and γ are numbers one knows that $(\alpha \times \beta) \times \gamma$ and $\beta \times (\alpha \times \gamma)$ result in the same number. We shall demand a similar property for products of values of quantities: If $q \in V_Q$ and $f \in V_F$ are values from the value spaces of Q and F and if α is a number the pairs of values $\langle \alpha q, f \rangle$ and $\langle q, \alpha f \rangle$ should determine the same value of $Q \otimes F$. Let us write the value of $Q \otimes F$ that is determined by a pair of values

$\langle q, f \rangle$ as $q \otimes f$. With this notation we can write the stated requirement formally as follows:

$$\forall(\alpha \in \mathbb{Q})\forall(q \in V_Q)\forall(f \in V_F): \quad (\alpha q)\otimes f = q \otimes (\alpha f) \tag{1.8.1}$$

In order to call $Q \otimes F$ a quantity we should have a definition of sum of values. Again we shall copy the properties that hold for products of numbers. We require two distributive laws:

$$\forall(r \in V_Q)\forall(q \in V_Q)\forall(f \in V_F): \quad (r+q)\otimes f = r\otimes f + q\otimes f \tag{1.8.2}$$

$$\forall(q \in V_Q)\forall(f \in V_F)\forall(g \in V_F): \quad q\otimes(f+g) = q\otimes f + q\otimes g \tag{1.8.3}$$

In the case that at least one of the quantities Q or F is one-dimensional the requirements (1.8.1), (1.8.2), and (1.8.3) already determine the sum of values uniquely. We shall discuss this simple case first. Let us assume that Q is one-dimensional. Let q and r be given values from V_Q and f and g given values from V_F. We want to determine the sum $(q\otimes f)+(r\otimes g)$. As Q is one-dimensional one can always write the value q as a multiple of the value r or the value r as a multiple of the value q. If both values are different from zero or both values are zero both alternatives exist. If one value is zero and the other is not zero only one of the alternatives applies. Without loss of generality we may assume that r can be written as a multiple of q. So we have $r = \alpha q$. Then the requirements (1.8.1) and (1.8.3) give

$$(q\otimes f)+(r\otimes g) = q\otimes(f+\alpha g) \tag{1.8.4}$$,

which determines the sum uniquely. So in the case that Q is one-dimensional the sum of values of $Q\otimes F$ has been determined by the algebraic structures of the spaces V_Q and V_F. With $\alpha = 0$ one concludes that for any value g the product $0\otimes g$ is the neutral element 0. And with the requirement (1.8.1) one concludes that $q\otimes 0$ is zero too. Further we shall demand that

$$\forall(q \in V_Q)\forall(f \in V_F): \quad (q\otimes f = 0 \;\Rightarrow\; (q = 0 \vee f = 0)) \tag{1.8.5}$$

With $q\otimes 0 = 0$ and with equation (1.8.4) we conclude that any value of the product quantity can be written in the form $\mathrm{U}\otimes f$ where U is some unit in the one dimensional values space V_Q. Then the requirement (1.8.5) allows to formulate the following criterion of equality of values:

$$\forall(f, g \in V_F): \quad (\mathrm{U}\otimes f = \mathrm{U}\otimes g \;\Leftrightarrow\; f = g) \tag{1.8.6}$$

Obviously the same arguments can be applied when F is one-dimensional.

In these cases one usually adopts the lazy way of writing products. We shall do the same: for linear quantities at least one being one-dimensional we define:

$$\left.\begin{array}{l}\text{For linear quantities } Q,\, F\\ \text{at least one being one-dimensional}\end{array}\right\}: \quad \left\{\begin{array}{l} QF \underset{Def.}{=} Q\otimes F \\ \forall(q \in V_Q)\forall(f \in V_F):\; qf \underset{Def.}{=} q\otimes f\end{array}\right.$$

(1.8.7)

If the two value spaces V_Q and V_F are different one can distinguish the elements of the ordered pair $\langle q, f \rangle$ by their individual nature so that it is not necessary to distinguish them by their location in the ordered pair. If V_Q and V_F are equal then any value of the quantity $Q \otimes F$ can be written in the form $(\alpha \times \beta)\mathrm{UU}$, where U is some unite in V_Q. This expression is completely symmetric. So in either case if $V_Q \neq V_F$ or if $V_Q = V_F$ it is not necessary to indicate which factor of a product stands to the left and which one stands to the right hand[1]. Therefore one defines for products of quantities where at least one factor is one-dimensional:

$$\left.\begin{array}{l}\text{For linear continuous quantities } Q, F \\ \text{at least one being one-dimensional}\end{array}\right\}: \quad \begin{cases} QF \underset{Def.}{=} FQ \\ \forall\left(q \in V_Q\right) \forall\left(f \in V_F\right): \; qf \underset{Def.}{=} fq \end{cases}$$

(1.8.8)

The case that both quantities Q and F are higher dimensional is quite more evolved. The requirements (1.8.9), (1.8.2), and (1.8.3) shall also be adopted in this case. But these requirements determine the sum $(q \otimes f) + (r \otimes g)$ only in the case when q and r are linearly dependent or f and g are linearly dependent. If both pairs are linearly independent a special definition of sum is required. In fact various types of product can be defined. For one type one even has to enlarge the value space of the quantity $Q \otimes F$ in order to accommodate the values $(q \otimes f) + (r \otimes g)$. We have already seen examples of enlargements of value spaces. In the original domain of the quantity mass there are no negative values. But in order to gain some mathematical simplicity one subjoins invented negative values. Later, in secondary usage of the mass quantity, the negative values even turn out to have a physical significance. The situation with products of multidimensional quantities is similar. In general there will be no object in the original domain of $Q \otimes F$ that corresponds to the values $(q \otimes f) + (r \otimes g)$ but with secondary usage of $Q \otimes F$ these values turn out to be useful. We shall postpone these definitions of products of multidimensional quantities to chapter 2 where multidimensional quantities are examined in detail so that the abstract definitions can be accompanied with intuitive examples.

Having a product of quantities one may think of defining the inverse operation; i.e. division. In order to do that, we need another extremely important notion. At the end of section 1.6 we mentioned that the students measured small mass values measuring the length of pieces of paper ribbon. That method presupposes that the mass values of these pieces are proportional to their length. In fact in their second experimental class these students received the specific task to correlate the length of pieces of copper wire (the kind used in transformers) and mass (measured with a fairly precise commercial balance). This was done with two types of copper wire and the data were plotted and fitted with two strait lines passing through the origin, which means the point $length = 0$ and $mass = 0$. The fitted lines define functions that map the value space of length V_d into the value space of mass V_m.

[1] Note that this has nothing to do with the fact that the quantum mechanical operators q and p do not commute. In fact, the product qp does not represent an observable.

The table 1.8.1 shows measured values for two types of copper wire, a thick one and a thin one, and Figure 1.8.1 shows the corresponding graphical representation of these data. Such representations are frequently used tools to represent, understand, and interpret measured data. Therefore a few words on such representations may be appropriate despite the fact that this issue is not within the scope of the present subject.

When you create a graph on a sheet of graph paper, never start by designing the data points! Start creating scales! Never use complicated scale factors! One unit of the represented quantity should correspond to a simple number of divisions of the graph paper. Simple numbers are 1, 2, 5 possibly multiplied by some integer power of 10. Use scale factors so as to take advantage of a large part of the paper. Next, write scale labels! These labels should have a reasonable and uniform density. The borders (axes) of the graph should show these labels but not the data values! The data should appear only as data points, thought there may exist exceptions if one wants to emphasize a special value. Declare the represented quantities on the axis! If the quantities are not number-valued declare the unit! Marc the data points with symbols that permit high precision and good visibility. Little dots are not appropriate because they may disappear under an interpolating curve.

Now let us come back to the issue of secondary quantities. The straight lines in figure 1.8.1 were drawn guided by an artistic view. Later we shall discuss more scientific methods to select a straight line that expresses a functional relationship in data sets. The interesting point is that our brain somehow detects that there is some simple relation of function type between the length values and mass values. A closer look reveals that not all data points are exactly on the line. But they are so close that we may assume that the deviations are due to experimental error and the straight lines represent a real physical law valid for copper wires used in transformers. The length and mass values are properties of the individual pieces of wire, whereas the function is a property of the type of wire. We could do this sort of experiment with many other types of copper wires and we would find similar results. With the help of an interpolating straight line we can define a function that maps the length value space of length V_d into the value space of mass V_m for every type of wire[2]. So for every type T of wire we get a function $\lambda_T : V_d \to V_m$ such that the value $\lambda_T(\ell)$ is the mass value of a piece of wire of type T and length ℓ.

These functions have a very special property: If we cut a piece of wire of a given type such that its length ℓ is exactly the sum of lengths ℓ_a and ℓ_b of two other pieces of the same type, the mass of that piece is the sum of the masses of the two pieces:

$$\ell = \ell_a + \ell_b \qquad \Rightarrow \qquad \lambda_T(\ell) \;=\; \lambda_T(\ell_a) + \lambda_T(\ell_b) \tag{1.8.10}$$

Also with multiples of lengths one has

$$\ell = \alpha\,\ell_0 \qquad \Rightarrow \qquad \lambda_T(\ell) = \alpha\;\lambda_T(\ell_0) \tag{1.8.11}$$

We may combine sum and multiplication with numbers to form linear combinations of values and express the properties (1.8.10) and (1.8.11) together by saying that one may form linear combinations before or after applying the function λ_T and one gets the

[2] If one uses the enlarged value spaces that include negative values the mappings are defined only on the subsets of positive values. But as we shall see later, they can be naturally extended to the extended values spaces.

same result. Functions that have this property are called ***linear functions*** or ***linear mappings***. This definition can be formulated for mappings between arbitrary linear spaces:

> Let A and B be two linear spaces with a commutative field $\mathbb{F}$ and let $f : A \to B$ be a mapping. f is called a **linear mapping** if and only if
>
> $$\forall(a,b \in A)\forall(\alpha,\beta \in \mathbb{F}): \quad f(\alpha a + \beta b) \;=\; \alpha f(a) \;+\; \beta f(b) \qquad (1.8.12)$$

Here we have applied a frequently used short had notation; instead of $\forall(a \in A)\forall(b \in A)$ be wrote a single quantifier for both variables; $\forall(a,b \in A)$.

Many other types of function are used in quantitative sciences but the linear functions outstandingly are the most important ones. The beginner should engrave this definition profoundly in his memory. The name “linear” stems from Latin “linea”, which means straight line, and which is related to “lignum”, which means wood – the trees tend to grow straight upward and the rulers used to be made of wood. Now, not every function that results in a straight line graph is linear in the sense of definition (1.8.12)! In fact many people call any function that result in a straight line graph a linear function. We shall not adhere to this nomenclature and shall restrict the name of linear function or linear mapping to the definition given above.

Table 1.8.1 Length and mass values of pieces of two types of copper wire.

Wire Type A		Wire Type B	
Length [cm]	Mass [g]	Length [cm]	Mass [g]
5.5	9.68	7.0	21.0
11.3	19.88	12.5	37.6
25.6	45.00	33.7	100
30.1	52.88	56.8	170
52.7	92.80	67.9	206
65.8	115.73	80.4	242
72.3	127.18	91.1	270
88.0	154.20	-	-
95.0	167.60	-	-
102.6	180.44	-	-

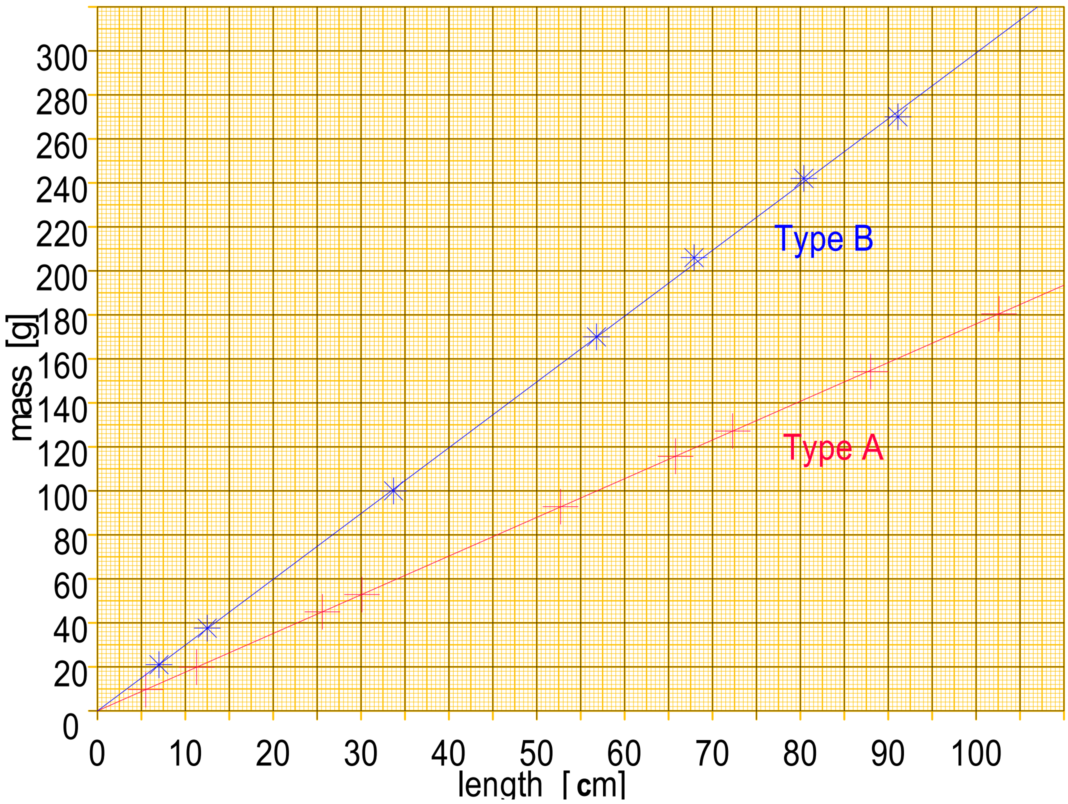

Figure 1.8.1 Graphical representation of the data of Tabel 1.8.1 and two linear functions that fit these data.

The linear functions λ_T are properties of the types of wire. Every type of wire has got its specific function, but several different types may have the same function. For instance, a wire with circular cross section and a wire with rectangular cross section represent different types of wire, but they may result in the same linear function. That looks like an equivalence relation on the set of types. Could it be that the totality of functions λ_T with their relationship with types forms a physical quantity whose domain is the set of types of wire? Every individual function λ_T could be a value of that quantity. What one needs in order to form a quantity is a definition of the sum of values. There is an obvious way of defining the sum of two functions: Let λ_T and λ_S be two functions that describe the correlation of mass and length values of copper wires of type T and S respectively. The sum of these functions is the function $(\lambda_T + \lambda_S)$ whose values are the sum of the values of λ_T and λ_S for all length values:

$$\forall(\ell \in V_d): \quad (\lambda_T + \lambda_S)(\ell) \underset{def.}{=} \lambda_T(\ell) + \lambda_S(\ell) \tag{1.8.13}$$

The functions λ_T and λ_S are linear. Obviously $(\lambda_T + \lambda_S)$ is linear as well. We may imagine that there exists a wire type with a relation of length and mass given by the function $(\lambda_T + \lambda_S)$. With this definition we may in fact consider the totality of these functions together with their relationship with types of wire a physical quantity. We shall call this quantity the *linear mass density*[3] and we represent it with the symbol λ.

[3] Here the word "linear" is not related to the definition (1.8.12). In this case it serves to distinguish the quantity λ from other types of density such as areal density and volumetric density.

As discussed in section 1.6 a definition of sum of values of a quantity entails a definition of multiplication of numbers and values of the quantity. So let λ_T be a linear density value of some wire type T and let α be a non-negative number. The linear density value $\alpha\lambda_T$ is the function such that for all length values ℓ one has

$$(\alpha\lambda_T)(\ell) \;=\; \alpha\big(\lambda_T(\ell)\big) \qquad (1.8.14)$$

If we combine (1.8.14) with the fact that λ_T is a linear function we get an interesting result:

$$(\alpha\lambda_T)(\ell) \;=\; \lambda_T(\alpha\,\ell) \qquad (1.8.15)$$

That equality tells us that an increase of density by a factor α has exactly the same effect on the mass value than an increase of length of a piece by the same factor α. What is interesting is a certain formal similarity of this result with the equation (1.8.1) that we demanded as a property of a product. Could it be that the application of the mapping λ_T on a length value has something to do with a multiplication of values?

We started with two quantities, the length *L* and the mass *M*, both defined on the set of pieces of copper wires. Then we formed a new quantity λ defined on the set of types of wire. We may still define another new quantity by multiplying the quantities length and linear density. Let us investigate the properties of the product λL.
Fist of all we have to enlarge the original domain of the quantity λ in order to define the product properly. The linear mass density was defined on the set of types of wires, whereas L was defined on the set of pieces of wire. Now every piece of wire P belongs to a certain type T_P and we may extend the domain of λ subjoining the set of pieces of wire and define the value of λ that can be attributed to the piece P in an obvious way: $\lambda_P \underset{def.}{=} \lambda_{T_P}$. With this precaution it makes sense to consider the product λL. Next we shall investigate the values of λL. In order to clarify the ideas first a note one nomenclature: The term $\lambda_T(\ell)$, which appears in the formula (1.8.14), is not a product! It is the value of the function λ_T at the point ℓ. To emphasize the difference of product and application of the function we shall exceptionally use the original product notation "$\otimes$" when we write a product of values.

Now let us imagine two pieces A and B of copper wire with linear densities λ_A and λ_B and lengths a and b respectively. We shall assume that these values are different from zero, which makes sense if we think of real wires. The cases with zero values can be treated separately. First let us suppose that these pieces have the same mass:

$$\text{hypothesis:} \qquad \lambda_A(a) = \lambda_B(b) \qquad (1.8.16)$$

We may write the length b as a multiple of the length a and apply the result (1.8.15):

$$\text{with } b = \alpha a \;:\; \lambda_A(a) = \lambda_B(\alpha a) = (\alpha\lambda_B)(a) \qquad (1.8.17)$$

This formula tells us that the functions λ_A and $\alpha\lambda_B$ coincide at the point *a*. As a is a non-zero element of a one-dimensional space every element of that space can be written as a multiple of *a*. Therefore, with the linearity of λ_A and $\alpha\lambda_B$, we can conclude from formula (1.8.17) that these functions coincide on the entire value space V_d. So they are the same function:

$$\lambda_A = \alpha \lambda_B \tag{1.8.18}$$

Now let us look at the values of the products $\lambda_A \otimes a$ and $\lambda_B \otimes b$. With $b = \alpha a$, with the rule (1.8.1) and with (1.8.18) we conclude:

$$\lambda_B \otimes b = \lambda_B \otimes (\alpha a) = (\alpha \lambda_B) \otimes a = \lambda_A \otimes a \tag{1.8.19}$$

So we have shown that $\lambda_A(a) = \lambda_B(b) \Rightarrow \lambda_A \otimes a = \lambda_B \otimes b$. Examining the steps carefully we notice that all arguments can also be stated in the inverse order so that one also has $\lambda_A \otimes a = \lambda_B \otimes b \Rightarrow \lambda_A(a) = \lambda_B(b)$. So one has

$$\lambda_A(a) = \lambda_B(b) \Leftrightarrow \lambda_A \otimes a = \lambda_B \otimes b \tag{1.8.20}$$

Obviously this equivalence is also valid when one or several of the involved values are zero. The mass values that result from the application of the functions λ_X on length values define the same equivalence classes on the set of pieces of wire than the values of the products $\lambda_A \otimes a$. Furthermore, the sum rule of products (1.8.2) and (1.8.3) are also identical with the rules (1.8.13) and (1.8.10). We may then identify the products $\lambda_A \otimes a$ with the values $\lambda_A(a)$.

When we defined the product of quantities we gave criteria to decide when two objects have the same value of the product quantity and when an object has a value of the product quantity that is the sum of values. But we actually never said what kind of mental objects shall represent the equivalence classes so that they can be called the values. Now we can use this lack of definiteness and chose the mass values $\lambda_A(a)$ as the mental objects that characterize the equivalence classes of the quantity $\lambda \otimes L$.

All what has been elaborated with the example of mass, length and linear density of pieces of wire could, exactly in the same fashion, be worked out with any linear mappings that map the value space V_Q of an arbitrary one-dimensional linear quantity Q into the value space V_F of an arbitrary (not necessarily one-dimensional) linear quantity F. But Q has to be one-dimensional. If Q is higher dimensional some details have to be modified and the products of higher dimensional quantities have to be used.

The totality R of linear mappings $R_T : V_Q \to V_F$ and their relation with the objects of the common domain of Q and F form a linear quantity. Any particular mapping R_T, where the index T refers to some type of objects (like the types of wires), is a value of that quantity. An object A with value Q_A has an F-value $F_A = R_{T_A}(Q_A)$, where T_A is the type to which the object A belongs. We may generically call such a quantity a Q-rate. So for example the linear density is a length rate. It describes the rate of mass increase with length increase. In chapter 2 we shall deal a lot with time-rates. For instance a temporal rate of dislocation in space is called *velocity*, a temporal rate of change of velocity is called *acceleration*.

Exactly with the same arguments that were used with the example of wires one can show that the application of an R_{T_A} on an element Q_A of V_Q can be identified with the multiplication of R_{T_A} and Q_A, So these linear mappings can all be written as products:

$$R_{T_A}(Q_A) \quad = \quad R_{T_A} \otimes Q_A \tag{1.8.21}$$

If one knows F_A, which is equal $R_{T_A}(Q_A)$, and Q_A then the formula (1.8.21) constitutes an equation with the unknown R_{T_A}. Due to the fact that Q is supposedly one-dimensional and that R_{T_A} is linear the mapping, R_{T_A} is uniquely determined, provided that $Q_A \neq 0$. That means if $Q_A \neq 0$ the equation (1.8.21) has a unique solution. That fact justifies to write the mapping R_{T_A} in the form

$$R_{T_A} = \frac{F_A}{Q_A} \tag{1.8.22}$$

As this formula is valid for all objects A from the intersection of the domains of the quantities F and Q we may state the result in terms of the quantities:

$$R = \frac{F}{Q} \tag{1.8.23}$$

So we have come to the division of quantities. The sequence of arguments seemed to be complicated. In fact, the times when we ate bananas on the trees and multiplied and divided quantities without knowledge of what we were doing were much easer. But the attempt to elevate these operations to the conscious level brings valuable insight into our way of being. The division of quantities is in fact another example of abstraction. When we introduce the quantity *linear density* we consider the mass and abstract or withdraw the aspect of length.

The formation of Q-rates in quantitative sciences is of such fundamental importance that we shall spend some more time and look at the quotient F/Q from an other point of view. We would like to write F/Q as a product of F and a sort of inverse of the quantity Q.

With the example of linear density we saw that linear mappings can be added in a natural way and that they form linear spaces. Associating objects with the elements of such spaces we get linear quantities. Let Q be a one dimensional linear quantity with domain D_Q and value space V_Q. The set of all linear mappings that map V_Q into the field of numbers is also a linear space. It shall be written as V_Q^* and it is called the *dual space of* V_Q.

Exercise: Show that the space V_Q^* is one-dimensional if V_Q is one-dimensional.

We may form quantities that use this space as its value space. We shall be interested in quantities with domains that are subsets of the domain D_Q. So let D_P be some subset of D_Q. Any mapping $P : D_P \to V_Q^*$ defines a quantity. Some of these quantities may be relevant and others may have no significance. Let P be such a quantity and let F be some linear quantity that has a domain with non-trivial intersection with D_P so that the product FP can be defined. The quantity P is one-dimensional and therefore we omitted the product symbol $\otimes$. As has been shown for any product that involves a one-dimensional quantity, any value of FP can be written as a product of values; *fp* with *f* from the value space V_F and p from V_Q^*. Such a value *fp* of the product quantity

FP can be interpreted as a linear mapping that maps the value space V_Q into the value space V_F. The application of fp on an element of V_Q is defined as follows:

$$\forall\left(q \in V_Q\right): \quad (fp)(q) \underset{def.}{=} f\, p(q) \tag{1.8.24}$$

On the right hand side the value f gets multiplied with the number $p(q)$.

Above we stated that not all quantities built from mappings $P: D_P \to V_Q^{\,*}$ may be relevant. But now we shall consider a specially important quantity of that kind. This quantity shall be written Q^{-1} or $1/Q$ and shall be called the inverse of Q. The domain of that quantity is the set $D_{Q^{-1}} = \left\{ x \in D_Q \,\middle|\, Q_x \neq 0 \right\}$. And it is defined as

$$\forall\left(A \in D_{Q^{-1}}\right): \quad Q_A^{\,-1}\left(Q_A\right) = 1 \tag{1.8.25}$$

With this quantity and with formula (1.8.24) one can now write the quotient (1.8.23) as a product:

$$R \;=\; F\, Q^{-1} \tag{1.8.26}$$

With a one-dimensional quantity Q one can define integer powers. For natural numbers one defies recursively:

$$Q^0 = 1 \tag{1.8.27}$$

$$\forall\left(n \in \mathbb{N}\right): \quad Q^{n+1} = Q^n\, Q \tag{1.8.28}$$

and

$$\forall\left(n \in \mathbb{N}\right): \quad Q^{-n} = \left(Q^n\right)^{-1} \tag{1.8.29}$$

Later we shall define also non-integer powers of one-dimensional quantities. The formulas (1.8.27), (1.8.28) and (1.8.29) refer to the quantities, analogous formulas hold for the values:

$$\forall\left(q \in V_Q\right): \quad q^0 = 1 \tag{1.8.30}$$

$$\forall\left(q \in V_Q\right)\forall\left(n \in \mathbb{N}\right): \quad q^{n+1} = q^n\, q \tag{1.8.31}$$

$$\forall\left(q \in V_Q\right)\forall\left(n \in \mathbb{N}\right): \quad q^{-n} = \left(q^n\right)^{-1} \tag{1.8.32}$$